POWER SUMS OF n AND $\varphi(n)$ INTEGERS

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POWER SUMS

Definition: Power sums for a positive integers n and $\varphi(n)$ are defined by

$$S_k(n) = 1^k + 2^k + \cdots + n^k, \quad k \in \mathbb{C};$$

$$\Psi_k(n) = a_1^k + a_2^k + \cdots + a_{\varphi(n)}^k, \quad a_i (i \in \mathbb{Z}) \in (\mathbb{Z}/n\mathbb{Z})^\times$$

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$k = 0$: $n, \qquad \varphi(n)$

$k = 1$: $\dfrac{n(n+1)}{2}, \qquad \dfrac{n\varphi(n)}{2}$

$k = 2$: $\dfrac{n(n+1)(2n+1)}{6}, \qquad \dfrac{\varphi(n)}{6}(2n^2 + \prod_{p|n} p)$

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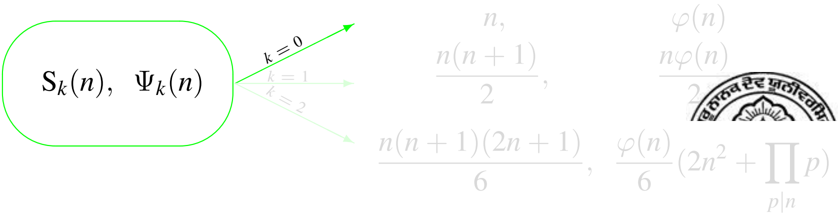

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Let $p,\ p_1,\ p_2, \cdots,\ p_r$ be distinct primes; $e,\ e_1,\ e_2,\ \cdots,\ e_r > 0$ and $k \geq 0$ be integers. Then

$$\Psi_k(p^e) = \mathbf{S}_k(p^e) - p^k\mathbf{S}_k(p^{e-1});$$

$$\Psi_k(n = p_1^{e_1}p_2^{e_2}) = \mathbf{S}_k(n) - p_1^k\mathbf{S}_k\left(\frac{n}{p_1}\right) - p_2^k\mathbf{S}_k\left(\frac{n}{p_2}\right) + p_1^kp_2^k\mathbf{S}_k\left(\frac{n}{p_1p_2}\right);$$

Theorem 1

$$\Psi_k(n) = \mathbf{S}_k(n) - \sum {p_i}^k\mathbf{S}_k\left(\frac{n}{p_i}\right) + \cdots + (-1)^r p_1^kp_2^k\cdots p_r^k\mathbf{S}_k\left(\frac{n}{p_1p_2\cdots p_r}\right),$$

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COROLLARIES

Cor. 1: $$\Psi_k(n) = \sum_{d|n} \mu(d) d^k \mathbf{S}_k\left(\frac{n}{d}\right); \quad \mathbf{S}_k(n) = n^k \sum_{d|n} \frac{\Psi_k(d)}{d^k}$$

where $\mu(d)$ is the Möbius function.

Cor. 2 $$\Psi_k(n) = \frac{n^{k+1}}{k+1} \sum_{m=0}^{[\frac{k}{2}]} C(k+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1})$$

where B_m is the $m-$th Bernoulli number defined via

$$\frac{y}{e^y - 1} = \sum_{m=0}^{\infty} B_m \frac{y^m}{m!};$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_{2m+1} = 0 \text{ for all } m = 1, 2, \cdots \ etc.$$

$\Psi_k(n)$

$$\Psi_0(n) = \frac{n^{0+1}}{0+1} \sum_{m=0}^{[\frac{0}{2}]} C(0+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1}) = \varphi(n);$$

$$\Psi_1(n) = \frac{n^{1+1}}{1+1} \sum_{m=0}^{[\frac{1}{2}]} C(1+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1}) = n\varphi(n)/2;$$

$$\Psi_2(n) = \frac{n^{2+1}}{2+1} \sum_{m=0}^{[\frac{2}{2}]} C(2+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1})$$

$$= n^2 \varphi(n)/3 + B_2\, n \prod_{p|n} (1 - p);$$

$\Psi_k(n)$

$$\Psi_3(n) = \frac{n^{3+1}}{3+1} \sum_{m=0}^{[\frac{3}{2}]} C(3+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1})$$

$$= n^3 \varphi(n)/4 + B_2 \frac{3n^2}{2} \prod_{p|n} (1-p);$$

$$\Psi_4(n) = n^4 \varphi(n)/5 + 2B_2 n^3 \prod_{p|n} (1-p) + B_4\, n \prod_{p|n} (1-p^3);$$

$$\Psi_5(n) = n^5 \varphi(n)/6 + B_2 \frac{5n^4}{2} \prod_{p|n} (1-p) + B_4 \frac{5n^2}{2} \prod_{p|n} (1-p^3)$$ & so on.

THE FUNCTIONAL EQUATIONS

Let x be a real variable and k be a nonnegative integer.

Definition: Define the power sum $S_k(x)$ *via* the generating function:

$$\frac{e^{(x+1)y} - e^{y}}{e^{y} - 1} := \sum_{k=0}^{\infty} \mathbf{S}_k(x) \frac{y^k}{k!}.$$

Remark 1:

$$\mathbf{S}_k(x)\big|_{\mathbb{Z}+} = \sum_{j=1}^{x} j^k;$$

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2. $\mathbf{S}_k(x) = (-1)^{k+1}\mathbf{S}_k(-1-x),\ k \neq 0$

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2. $$\frac{e^{(x+1)y} - e^y}{e^y - 1} = -\frac{e^{(1-1-x)(-y)} - e^{-y}}{e^{-y} - 1} - 1.$$

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$$\Psi_k(x,n) = \sum_{d|n} \mu(d) d^k \mathbf{S}_k\left(\frac{x}{d}\right).$$

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Definition 1: Call a representation of a real number r its simplest form if and only if it contains at most one rational term. This rational term will be called as the rational part of r, and we shall denote it by $\mathbf{Q}(r)$.
Definition 2: Call $x = 0, -1$ as *trivial zeros* of $\mathbf{S}_k(x)$ for $k = 2, 3, 4, \ldots$ each with multiplicity at most 2.

$r = 3 + \sqrt[3]{5},\ 1/2 + (5/2 + \sqrt[3]{5}),\ -2 + 1 + (4 + \sqrt[3]{5}); \mathbf{Q}(r) = 3$

Conjecture

Let x be a nontrivial zero of $\mathbf{S}_k(x)$ for a fixed $k > 3$, then real(x) in its simplest form has rational part $\mathbf{Q}(x) = -\frac{1}{2}$.

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THE ALTERNATING POWER SUMS

Definition: Call $\psi_k(p)$ the $k-$th Euler polynomial (see Ramanujan [9])defined by

$$\frac{1}{e^y + p} = \sum_{k=0}^{\infty} \frac{(-1)^k \psi_k(p)}{(p+1)^{k+1}} \frac{y^k}{k!}, \; p \neq -1, \; |y| < |\log(p)|.$$

Theorem 4(Ramanujan)

For $|p| < 1$ and $k \geq 0$,

$$(p+1)^{-1-k} \psi_k(p) = \sum_{n=0}^{\infty} (-p)^n (n+1)^k.$$

THE ALTERNATING POWER SUMS

Definition: Let $\alpha \neq 0$. We say $\mathbf{E}_k(\alpha)$ is the $k-$th α-Euler number defined via the generating function

$$\frac{\alpha}{\alpha e^y - 1} = \sum_{k=0}^{\infty} \mathbf{E}_k(\alpha) \frac{y^k}{k!}, \ \alpha \neq 1.$$

$$\mathbf{E}_0(\alpha) = \frac{\alpha}{\alpha - 1}; \ \mathbf{E}_1(\alpha) = -\frac{\alpha^2}{(1-\alpha)^2}$$

$$\mathbf{E}_2(\alpha) = \frac{\alpha^2(\alpha+1)}{(\alpha-1)^3}; \ \mathbf{E}_3(\alpha) = -\frac{\alpha^2(1+4\alpha+\alpha^2)}{(\alpha-1)^4} \ etc.$$

THE ALTERNATING POWER SUMS

$$E_k(-1) = (1-2^k)\frac{B_{k+1}}{k+1};$$

$$\frac{\alpha}{\alpha e^y - 1} = \frac{1}{e^y + (-\alpha^{-1})} \Rightarrow (-1)^k \psi_k(-\alpha^{-1})/(1-\alpha^{-1})^{k+1} = \mathbf{E}_k(\alpha).$$

For $\alpha \neq 0, \ 1$. Then

Theorem 5

$$\mathbf{E}_{k+1}(\alpha) = \alpha^2 \frac{\partial}{\partial \alpha}(\mathbf{E}_k(\alpha)/\alpha)$$

$$\alpha^{-1}\mathbf{E}_k(\alpha) = (-1)^{k+1}\alpha \mathbf{E}_k(\alpha^{-1}), \ k \neq 0.$$

THE ALTERNATING POWER SUMS

Definition: Call $\mathbf{S}_k(\alpha, x)$ as the $\alpha-$power sum defined by

$$\frac{\alpha^{x+1}e^{(x+1)y} - \alpha e^y}{\alpha e^y - 1} = \sum_{k=0}^{\infty} \mathbf{S}_k(\alpha, x)\frac{y^k}{k!}, \ \alpha \neq 1$$

1. $\mathbf{S}_0(\alpha, x) = \alpha\dfrac{1-\alpha^x}{1-\alpha}$, $\mathbf{S}_k(\alpha, 1) = \alpha$, $\lim_{\alpha\to 1-} \mathbf{S}_k(\alpha, x) = \mathbf{S}_k(x)$.
2. When n is a positive integer
$$\mathbf{S}_k(\alpha, n) = \alpha + \alpha^2 2^k + \alpha^3 3^k + \cdots + \alpha^n n^k.$$
3. From definition
$$\mathbf{S}_k(\alpha, x) = \sum_{m=0}^{k} C(k, m)\mathbf{E}_m(\alpha)\{\alpha^x(1+x)^{k-m} - 1\}, \ \alpha \neq 1$$

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THE ALTERNATING POWER SUMS

For $k > 0$ $\mathbf{S}_k(\alpha, -1) = 0$ and $\mathbf{S}_0(\alpha, -1) = -1$, this leads to the identity

$$\sum_{m=0}^{k} C(k,m)\mathbf{E}_m(\alpha) = \begin{cases} \alpha^{-1}\mathbf{E}_k(\alpha), & \text{for } k \neq 0 \\ \mathbf{E}_0(\alpha), & \text{for } k = 0. \end{cases} \quad (1)$$

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$$\mathbf{S}_{k+1}(\alpha, x) = \alpha \frac{\partial}{\partial \alpha} \mathbf{S}_k(\alpha, x)$$

$$\frac{\alpha^{x+1}e^{(x+1)y} - \alpha e^y}{\alpha e^y - 1} = -\frac{(\alpha^{-1})^{1-1-x}e^{(1-1-x)(-y)} - \alpha^{-1}e^{-y}}{\alpha^{-1}e^{-y} - 1} - 1 \Rightarrow$$

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ABEL–SUM

As $\mathbf{S}_0(\alpha, x) = \alpha \dfrac{1-\alpha^x}{1-\alpha}$, and for $|\alpha| < 1$ $\alpha^x \to 0$ as $x \to \infty$, therefore $\mathbf{S}_0(\alpha, x) \to \dfrac{\alpha}{1-\alpha} = -\mathbf{E}_0(\alpha)$ as $x \to \infty$. For $k > 0$ and $|\alpha| < 1$,

$$\lim_{x\to\infty} \mathbf{S}_k(\alpha, x) = \lim_{x\to\infty} \sum_{m=0}^{k} C(k,m)\mathbf{E}_m(\alpha)\{\alpha^x(1+x)^{k-m} - 1\}$$

$$= -\sum_{m=0}^{k} C(k,m)\mathbf{E}_m(\alpha) = -\alpha^{-1}\mathbf{E}_k(\alpha).$$

The Abel sum for the alternating series $\sum_{n=1}^{\infty}(-1)^n n^k$ is now given by

$$\lim_{\alpha\to -1^+} \sum_{n=1}^{\infty} \alpha^n n^k = \lim_{\alpha\to -1^+} (-\alpha^{-1}\mathbf{E}_k(\alpha)) = \mathbf{E}_k(-1).$$

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$$\begin{aligned}\lim_{x\to\infty} \mathbf{S}_k(\alpha, x) &= \lim_{x\to\infty} \sum_{m=0}^{k} C(k,m)\mathbf{E}_m(\alpha)\{\alpha^x(1+x)^{k-m} - 1\} \\ &= -\sum_{m=0}^{k} C(k,m)\mathbf{E}_m(\alpha) = -\alpha^{-1}\mathbf{E}_k(\alpha).\end{aligned}$$

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ABEL–SUM

Let s be a positive integer and ω_s denotes an $s-$th root of unity. Then

$$\frac{1}{(\alpha e^{y/s})^s - 1} = \frac{1}{s}\left\{\frac{1}{\alpha e^{y/s} - 1} + \frac{1}{\alpha\omega_s e^{y/s} - 1} + \cdots + \frac{1}{\alpha\omega_s^{s-1} e^{y/s} - 1}\right\}$$

$$\mathbf{E}_k(\alpha^s) = \frac{\alpha^{s-1}}{s^{k+1}}\{\mathbf{E}_k(\alpha) + \omega_s^{-1}\mathbf{E}_k(\alpha\omega_s) + \cdots + \omega_s^{-(s-1)}\mathbf{E}_k(\alpha\omega_s^{s-1})\}$$

$$\mathbf{E}_k(\alpha^2) = \frac{\alpha}{2^{k+1}}(\mathbf{E}_k(\alpha) - \mathbf{E}_k(-\alpha)).$$

Definition: $\mathbf{E}_k(1) := \frac{1}{1-2^{k+1}}\mathbf{E}_k(-1)$

$$\mathbf{E}_k(1) = \frac{1-2^k}{1-2^{k+1}}\frac{B_{k+1}}{k+1} = \frac{1-2^k}{1-2^{k+1}}\zeta(-k)$$

EXTENDING DEFINITION OF $\mathbf{S}_k(\alpha, x)$

Note that for a positive integer n $\lim_{\alpha\to 1^-} \lim_{n\to\infty} \mathbf{S}_k(-\alpha, n) =$ Abel sum $-\eta(-k)$ for the divergent series $-1 + 2^k - 3^k + \cdots \infty$.

$$\mathbf{S}_k(\alpha, x) := \int_0^\alpha t^{-1}\mathbf{S}_{k+1}(t, x)dt, \; k = 0, \pm 1, \pm 2 \dots$$

$$\lim_{x\to\infty} \mathbf{S}_{-1}(\alpha, x) = -\log|\alpha - 1|$$

$$\lim_{x\to\infty} \mathbf{S}_{-2}(\alpha, x) = -\log(\alpha)\log(1-\alpha) + \zeta(2) - \lim_{x\to\infty} \mathbf{S}_{-2}(1-\alpha, x),$$

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$$\lim_{x\to\infty} \mathbf{S}_{-2}(1/2, x) = -\frac{(\log(2))^2}{2} + \frac{1}{2}\zeta(2)$$

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EXTENDING DEFINITION OF $\mathbf{S}_k(\alpha, x)$

$$\lim_{x\to\infty} \mathbf{S}_{-3}(\alpha,x) = \frac{1}{2}\int_0^{\alpha} \frac{(\log(t))^2}{1-t}dt + \log(\alpha)\lim_{x\to\infty}\mathbf{S}_{-2}(\alpha,x) - \frac{(\log(\alpha))^2}{2}\lim_{x\to\infty}\mathbf{S}_{-1}(\alpha,x)$$

Theorem 7

For $0 < \alpha < 1$ and $k = 1, 2, 3, \ldots$

$$\lim_{x\to\infty} \mathbf{S}_{-k}(\alpha,x) = \frac{(-1)^{k-1}}{(k-1)!}\int_0^{\alpha} \frac{(\log(t))^{k-1}}{1-t}dt - \sum_{\beta=1}^{k-1}(-1)^{\beta}\frac{(\log(\alpha))^{\beta}}{(\beta)!}\lim_{x\to\infty}\mathbf{S}_{-k+\beta}(\alpha,x) \quad (2)$$

GENERALIZED POWER SUMS

We are interested in the case when $|a| < 1$ and $x \to \infty$. So, for $|a| < 1$, let us denote $\lim_{x\to\infty} \mathbf{S}_k(a,b,c,x)$ by $\mathbf{S}_k(a,b,c,\infty)$.

Definition: Define the generalized power sum $\mathbf{S}_{-k}(a,b,c,\infty)$ for $k = 0, \pm 1, \pm 2, \pm 3 \ldots$ and $c \neq 0$, $a \neq 0, 1$, $\text{real}(bc^{-1}) > 0$ via the functional equations:

$$(1)\ \mathbf{S}_0(a,b,c,\infty) = \frac{a}{1-a},$$

$$(2)\ (-k+1)\mathbf{S}_{-k}(a,b,c,\infty) = \frac{\partial}{\partial b}\mathbf{S}_{-k+1}(a,b,c,\infty),$$

$$(3)\ \mathbf{S}_{-k}(a,b,c,\infty) = \frac{c^{-1}}{a^{(bc^{-1}-1)}}\int_0^a t^{(bc^{-1}-2)}\mathbf{S}_{-k+1}(t,b,c,\infty)\,dt.$$

GENERALIZED POWER SUMS

Definition:

$$\frac{a^{1+x}e^{(b+cx)y} - ae^{by}}{ae^{cy} - 1} := \sum_{k=0}^{\infty} \mathbf{S}_k(a,b,c,x)\frac{y^k}{k!}, \; a,b,c \neq 0, \; a \neq 1 \tag{3}$$

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