Roll No...............

Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Final Examination 2010 for M.Sc.(Maths) Semester-III

Course name: Topology — **Max. Marks 100**
Course code: Math 532 — **Time Allowed: 03 Hours**

Note: Attempt any *ten* questions selecting two questions from each of the sections A, B, C, D, & E.

Section A

1. Let X be a topological space and $S \subset X$. Define $\partial S := \{x \in X \mid U \cap S \neq \emptyset \text{ and } U \cap (X \sim S) \neq \emptyset\}$ for every open subset $U \ni x$ of X. Show that

   (i) $\partial S = \partial(X \sim S)$ and ∂S is closed in X

   (ii) $\bar{S} = S \cup \partial S$

   (iii) $\text{Int}(S) = S \sim \partial S$

   (iv) $X \sim \text{Int}(S) = \overline{X \sim S}$

   (v) $X \sim \bar{S} = \text{Int}(X \sim S)$ (5×2)

2. (a) Let S be a dense subset of a topological space X. Then prove that for every open subset U of X, $\overline{S \cap U} = \bar{U}$. (5)

   (b) Let $S \subset \mathbb{R}$ in the order topology of $\mathbb{R}$. Let $x \in \mathbb{R}$; prove that $x \in \bar{S}$ if and only if for each positive integer n there is a real number $x_n \in S$ such that $|x_n - x| < \frac{1}{n}$. (5)

3. (a) Let A be a subset of a topological space X. Then prove that $x \in \bar{A}$ if and only if every basis element B containing x intersects A. (5)

   (b) Prove that every finite point set in a Hausdörff space is closed. (5)

4. Assume that the real line $\mathbb{R}$ is connected. Prove that the countable product $\mathbb{R}^\omega$ of $\mathbb{R}$ with itself is connected in the product topology but $\mathbb{R}^\omega$ is not connected in the box topology. (10)

Section B

5. Prove that a function f from a topological space X to a topological space Y is continuous if and only if for every open subset V of Y there is an open subset U of X such that $f(U) \subset V$. Hence show that the function $f : \mathbb{R} \to \mathbb{R}$ in the order topology defined by

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   is continuous exactly at one point. (10)

6. Let A be a subset of a topological space X and Y be a Hausdörff space such that the map $f : A \to Y$ is continuous. Then prove that if f may be extended to a continuous map $g : \bar{A} \to Y$ then g is uniquely determined by f. (10)

7. Consider the countable product $\mathbb{R}^\omega$ of the real line with itself. Define a map $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t,\ t,\ t,\ \cdots)$. Show that f is continuous if $\mathbb{R}^\omega$ is given the product topology but f is not continuous if $\mathbb{R}^\omega$ is given the box topology. (10)

8. (a) Let $f : X \to Y$ be a continuous map. Let x is a limit point of a subset $A \subset X$. Is it necessarily true that $f(x)$ is a limit point of $f(A)$? Give an appropriate reason in support of your answer. (5)

   (b) Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subset X_\alpha$ for each α. Consider the product $X = \prod_{\alpha \in J} X_\alpha$. Prove that $\overline{\prod_{\alpha \in J} A_\alpha} = \prod_{\alpha \in J} \bar{A}_\alpha$ in the product topology of X. (5)

Section C

9. Prove that a topological space X is normal if and only if for every pair of disjoint closed subsets A and B of X there is a continuous map $f : X \to [0,1] \subset \mathbb{R}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. (10)

10. Let X be a normal space and A be a closed subspace of X. Let $f : A \to [-1,1]$ be a continuous map from A to the closed interval [-1,1] of $\mathbb{R}$. Then prove that f may be extended to a continuous map $g : X \to [-1,1]$. (10)

11. Prove that a topological space is regular if and only if for every point $x \in X$ and an open subset $U \ni x$ of X there is an open set $V \ni x$ such that $\bar{V} \subset U$. (10)

12. Let $\{X_n\}_{n \in \mathbb{Z}^+}$ be a family of completely regular spaces. Prove that the product space $\prod_{n \in \mathbb{Z}^+} X_n$ is completely regular in the product topology. (10)

Section D

13. Consider the subset $A = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\}$ of the real line $\mathbb{R}$. Prove that for every covering of A by sets open in $\mathbb{R}$ there is a finite sub collection still covering A. (10)

14. Let X be a simply ordered set having the greatest lower bound property. Prove that in the order topology, each closed interval in X is compact. (10)

15. Prove that a continuous bijective map from a compact topological space to a Hausdörff space is always a homeomorphism. (10)

16. Let $\mathscr{A}$ be an open covering of the metric space(X,d). If X is compact prove that there is a real number $\delta > 0$ such that for every subset of X having diameter less than δ, there is an element of $\mathscr{A}$ containing it. (10)

Section E

17. Prove that if a topological space X is compact then it is limit point compact but the converse may not be always true. (10)

18. Prove that if a non compact topological space X is locally compact Hausdörff then there is a one point compactification of X. (10)

19. Let X be a Hausdörff space. Prove that X is locally compact if and only if for any $x \in X$ and a neighborhood U of x there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$. (10)

20. (a) Let $\{C_i\}_{i\in\mathbb{Z}^+}$ be a sequence of closed sets in a compact topological space X satisfying $C_1 \supset C_2 \supset \cdots C_i \supset C_{i+1} \cdots$. Prove that the intersection $\bigcap_{i\in\mathbb{Z}^+} C_i$ is nonempty. (5)

    (b) Let X be countably compact topological space and A be a subset of X having no limit point. Prove that A is finite. (5)