Department of Mathematics

Guru Nanak Dev University, Amritsar-143005

Notes on 'Definite integrals' & Assignment#8

In these notes we will define definite integral of a real function of one and two real variables as limits of finite sums.

Definition: Let $f : [a,b] \to \mathbb{R}$ be a function. We define a partition of $[a,b]$ to be the set of smaller subintervals $[x_i, x_{i+1}]$ for $i = 0, 1, ..., n-1$ with $x_0 = a$ and $x_n = b$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. If length of each subinterval is assumed to be same say h then $h = \dfrac{b-a}{n}$ and $x_i = a + ih$.

Definition: Let $f : [a,b] \to \mathbb{R}$ be a map and $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$. Define following finite sums:

$$s_n = \sum_{i=0}^{n-1} \inf_{[x_i, x_{i+1}]} \{f(x)\}(x_{i+1} - x_i); S_n = \sum_{i=0}^{n-1} \sup_{[x_i, x_{i+1}]} \{f(x)\}(x_{i+1} - x_i)$$

If $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} S_n$ exist and are equal, we say that f is integrable over $[a,b]$ and the definite integral of f over $[a,b]$ is defined as

$$\int_a^b f(x)\, dx := \lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n.$$

Definition (Limit of Sum): Define for a partition as above of interval $[a,b]$ the following finite sum:

$$\sum := \lim_{h\to 0} h\{f(a) + f(a+h) + \cdots + f(a+nh)\}.$$

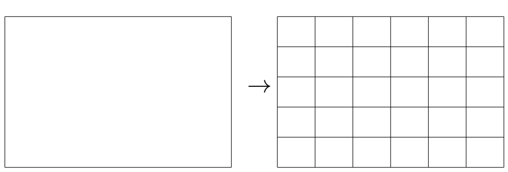
If the above limit exists then we say that definite integral of f over the interval $[a,b]$ exists and is denoted by $\int_a^b f(x)dx$ or $\int_a^b f$ or simply $\int_{[a,b]} f$ and is defined by

$$\int_a^b f(x)dx := \lim_{h\to 0} h\{f(a) + f(a+h) + \cdots + f(a+nh)\}$$

Theorem (Fundamental Theorem of Calculus): *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. If there exists a function $g(x)$ such that $g'(x) = f(x)$ then $\int_a^b f(x)dx = g(b) - g(a)$.*

The fundamental theorem of calculus establishes a connection between the definite integral and the antiderivative.

Definition (Partition of a rectangle): Let $R := [a,b] \times [c,d] \subset \mathbb{R}^2$ be a rectangular region. A partition P of R is defined as $P := P_1 \times P_2$ where $P_1 : a = x_0 < x_1 < \cdots < x_m = b$ and $P_2 : c = y_0 < y_1 < \cdots < y_n = d$ are partitions of the intervals $[a,b]$ and $[c,d]$ respectively.



R=[a,b]× [c,d]   Partition of R

Double Integral: For a fixed j consider the volume of the surface defined by the function $f : R \to \mathbb{R}$ as sum of the volumes of the m parallelepiped regions with base at rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}],\ 0 \le i \le m-1$

$$v_j = (y_{j+1} - y_j) \sum_{i=0}^{m-1} \inf_{x\in[x_i, x_{i+1}]} \{f(x, y_j)\}(x_{i+1} - x_i) \tag{0.1}$$

$$V_j = (y_{j+1} - y_j) \sum_{i=0}^{m-1} \sup_{x\in[x_i, x_{i+1}]} \{f(x, y_j)\}(x_{i+1} - x_i) \tag{0.2}$$

If the function $f(x, y_j)$ is integrable over $[a,b]$ then

$$\lim_{m\to\infty} v_j = \lim_{m\to\infty} V_j = (y_{j+1} - y_j) \int_a^b f(x, y_j)dx$$

Now summing up these volumes for each $j = 1, 2, ..., n$ we obtain

$$s_n = \sum_{j=0}^{n-1} \inf_{y\in[y_j, y_{j+1}]} \left\{\int_a^b f(x,y)dx\right\}(y_{j+1} - y_j) \tag{0.3}$$

$$S_n = \sum_{j=0}^{n-1} \sup_{y\in[y_j, y_{j+1}]} \left\{\int_a^b f(x,y)dx\right\}(y_{j+1} - y_j). \tag{0.4}$$

We say that the double integral of f over the rectangular region R is defined if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n$, and then Eqs.(0.3)-(0.4) reduce to $S = \int_c^d \left(\int_a^b f(x,y)dx\right) dy$. S is called the double integral of f over R and we write $\int_R f dR := \int_c^d \left(\int_a^b f(x,y)dx\right) dy$. Changing the role of x and y in the above discussion we see that

$$\int_c^d \left(\int_a^b f(x,y)dx\right) dy = \int_a^b \left(\int_c^d f(x,y)dy\right) dx$$

so we note that *the order of integration is immaterial for rectangular domains.*

Remark 1: If the domain R is not rectangular i.e. say $R := \{(x,y) \mid c \le y \le d,\ \varphi_1(y) \le x \le \varphi_2(y)\}$ even then the above calculations are valid and here

$$\int_R f dR := \int_c^d \left(\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)dx\right) dy.$$

However here unlike the rectangular domains the order of integration can not be changed simply by interchange of limits but one needs to evaluate the limits explicitly from the geometry of R.

Remark 2: If we take $f(x,y) := 1$ then for any region R, $\int_R f dR := \int_R dR$ which gives the area of the region R. As an application of this we prove that area of a circle of radius r is πr^2 by choosing R bounded by the circle $x^2 + y^2 = r^2$ and $f(x,y) = 1$ such that

$$\int_{x^2+y^2\le r^2} 1 dR := \int_{-r}^{r} \left(\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\right) dx = 4\int_0^r \sqrt{r^2 - x^2} dx.$$

Remark 3: Some times it is easy to evaluate the double integrals using change of variables for if under a continuous nonsingular transformation of coordinates $(x,y) \to (u,v)$ if the region R is transformed to a region R' then

$$\boxed{\int_R f(x,y)dxdy = \int_{R'} f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)} dudv}$$

Example: To calculate the volume of the sphere $x^2+y^2+z^2 = a^2$, for upper hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ defines a function of two variables with domain R bounded by the circle $C = x^2 + y^2 = a^2$ where $-a \le x \le a$ and $-\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}$. $\therefore$ the volume of the sphere $=2\times \int_R \sqrt{a^2 - x^2 - y^2} dR$ which simplifies to

$$V = 2\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\ dxdy.$$

Using $x = r\cos\theta,\ y = r\sin\theta,\ \frac{\partial(x,y)}{\partial(r,\theta)} = r$ and here $0 \le r \le a,\ 0 \le \theta \le 2\pi$ s.t.

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\ dxdy = 8\int_0^a r\sqrt{a^2 - r^2}\ dr \int_0^{\pi/2} d\theta$$

which sums to $\dfrac{4}{3}\pi a^3$.

[1]E-mail: sonumaths@gmail.com; Web page: https://sites.google.com/site/sonumaths2/

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Assignment#8 on 'Definite integrals'

1. Evaluate the following definite integral as a limit of sum and verify the fundamental theorem of calculus:

   (a) $\int_0^1 e^{2x}$ (b) $\int_0^{\pi/2} \sin ax\ dx$ (c) $\int_0^{2\pi} |\sin x| dx$ (d) $\int_{-2}^{6} |x|\ dx$ (e) $\int_{-1}^{1} x^2 e^x\ dx$

2. Evaluate the following definite integrals:

   (a) $\int_0^{\pi/2} \log(\sin x)\ dx$ (b) $\int_0^{\pi/2} \log(\cos x)\ dx$ (c) $\int_0^{\pi/2} \frac{dx}{(a\sin x + b\cos x)^2}$ (d) $\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$

3. Derive a reduction formula for $\int x^n e^{ax} dx$. Hence evaluate $\int_0^1 x^{10} e^x dx$.
4. Evaluate the following double integrals:

   (a) $\int_{[0,1]\times[0,\pi/2]} x \log(\sin y)\ dxdy$ (b) $\int_0^1 \int_x^{x^2+1} \frac{y}{x+y}\ dxdy$ (c) $\int_0^{\pi/2} \int_0^{\sqrt{1-y^2}} x\ dxdy$

5. $\int_R \frac{xy}{\sqrt{x^2+y^2}}\ dR$ where R is the region bounded by the curves $x^2+y^2=1$ and $y=x$.
6. Evaluate the double integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 y^2 e^{-x^2-y^2} dxdy$ by change of variables $(x,y)\to(r,\theta)$ where the (r,θ) are plane polar coordinates.
7. Evaluate volume of the tetrahedron bounded by the planes $x+y+z=2,\ x=1,\ y=0,\ z=0$.
8. Calculate the volume of the right circular cone given by $x^2+y^2-\mu^2 z^2=0$ with the base defined by the circular region $x^2+y^2\le r^2$.
9. Calculate $\int_R (x+y)dxdy$ where $R := \{(x,y) \mid x\ge 0,\ y\le 1,\ y\ge x^2\}$. Verify that the change of order gives the same value of the integral.
10. Consider $z=f(x,y)$ defined over the surface of the sphere $S := \{(x,y,z) \mid x^2+y^2+z^2=r^2\}$. Subdivide S into small curved rectangles and consider one representative with area $dS\hat{n}$ with unit normal $\hat{n}$. Then projection of this area element over $xy-$plane is just $dxdy\hat{k}$ such that $dS\hat{n}.\hat{k}=dxdy$ i.e. $\boxed{dS=\frac{dxdy}{\hat{n}\cdot\hat{k}}}$. If projection of S onto $xy-$plane is R, establish that

$$\int_S f(x,y)dS = \int_R f(x,y)\frac{dxdy}{\hat{n}\cdot\hat{k}}.$$

11. Using the analysis of the preceding problem, prove that the surface area of a sphere of radius r is $4\pi r^2$.
12. Evaluate the volume and surface area of the Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

*13. Let $M(x,y)$ and $N(x,y)$ be two smooth functions defined in and on closed curve in $xy-$plane bounding a surface S. Prove that

$$\int_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dxdy = \oint_C (Mdx+Ndy).$$

[Comment: *This statement is called Green's theorem in plane. You should have some knowledge of line integrals in order to prove it.*]

*14. Let $\mathbf{F}$ be a smooth vector point function defined in and on an open surface S bounded by a closed curve C. Then prove that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}dS$$

where $\hat{\mathbf{n}}$ is a unit normal to the surface. This statement is called the Stokes theorem.

*15. Let $\mathbf{F}$ be a smooth vector point function defined in and on a closed surface S bounding a volume V. Then prove that

$$\int_S \mathbf{F}\cdot dS\hat{\mathbf{n}} = \int_V (\nabla\cdot\mathbf{F})dV.$$

This statement is called the Gauss divergence theorem. You need to know triple integrals first to prove this theorem.

[1]E-mail: sonumaths@gmail.com; Web page: https://sites.google.com/site/sonumaths2/