Department of Mathematics

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Notes on 'Antiderivatives' & Assignment#7

In these notes we will define an antiderivative of a real function of one real variable and discuss various techniques of obtaining the antiderivative.

Definition: Let $f : [a, b] \to \mathbb{R}$ be a function. We say that antiderivative of f exists at a point $x \in (a, b)$ if there is a differentiable function $g : (a, b) \to \mathbb{R}$ such that $g'(x) = f(x)$. If antiderivative of f exist at x we say that $g(x)$ is the antiderivative of f at x and write $g(x) = \int f(x)dx$.

Example: We know that $(\sin x)' = \cos x$ for all $x \in [0, 2\pi]$ therefore by definition, $\sin x$ is an antiderivative of $\cos x$ at x and we write $\int \cos x \; dx = \sin x$.

Remark: If $g'(x) = f(x)$ then for any constant real c, $(g(x) + c)' = g'(x) + c' = g'(x) = f(x)$; it follows that if $g(x)$ is an antiderivative of f at x then so is $g(x)+c$. We may write $\int f(x) \; dx = g(x) + c$.

Theorem 1: *Let $f_1(x)$ and $f_2(x)$ possess antiderivatives at a point x. Then $f_1(x) + f_2(x)$ also possess antiderivative at x. More precisely* $\int (f_1(x) + f_2(x))dx = \int f_1(x)dx + \int f_2(x)dx$.

Proof: Let $g_1'(x) = f_1(x)$ and $g_2'(x) = f_2(x)$ then observe that $(g_1(x) + g_2(x))' = g_1'(x) + g_2'(x) = f_1(x) + f_2(x)$. This means $g_1 + g_2$ is the antiderivative of $f_1 + f_2$ at x. This proves that theorem.

Theorem 2(Change of Variable): *Let $g'(x) = f(x)$ and $x : (a, b) \to (c, d)$ is a differentiable function of t such that $\frac{dx}{dt} \neq 0$ and is finite. Then* $\int f(x)dx = \int (f \circ x)(t)\frac{dx}{dt}dt$.

Proof: Since $f(x) = \frac{d}{dx}g(x(t)) = \frac{d}{dt}(g \circ x)(t)\frac{dt}{dx}$. This gives us $\frac{d}{dt}(g \circ x)(t) = (f \circ x)(t)\frac{dx}{dt}$ because $\frac{dt}{dx} \neq 0$. Result follows now from definition of antiderivative.

Theorem 3(Integration by Parts): *Let f and g be two functions of x with antiderivatives at point x and $f'(x)$ exists. Then the map $f(x)g(x)$ has an antiderivative at x given by*

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int \left(f'(x)\int g(x)dx\right)dx.$$

Proof: As $\frac{d}{dx}\left(f(x)\int g(x)dx\right) = f'(x)\int g(x)dx + f(x)g(x)$; the assertion follows from definition of antiderivative and theorem 1.

Definition(Definite Integral): Let $f : [a, b] \to \mathbb{R}$ be a map with antiderivative $g(x)$ at each point $x \in [a, b]$. Then definite integral of f over the interval $[a, b]$ or (a, b) or $[a, b)$ or $(a, b]$ is defined as $\int_a^b f(x)dx := g(b) - g(a)$.

1. Obtain the antiderivative of each of the following functions:

   (a) $\tan x$
   (b) $\dfrac{a\sin x - b\cos x}{a\sin x + b\cos x}$
   (c) $\dfrac{1}{(x-1)^2(x-3)}$
   (d) $\dfrac{1}{2+x^2}$
   (e) $\dfrac{1}{2-x^2}$
   (f) $\dfrac{1}{\sqrt{a^2-x^2}}$
   (g) $\sec 2x$
   (h) $\dfrac{2x+3}{x^2+3x+2}$
   (i) $\dfrac{f'(x)}{\{f(x)\}^n}$
   (j) $\dfrac{\sin x}{\sin(x+a)}$
   (k) $\dfrac{1}{1+x^2}$
   (l) $\dfrac{1}{\sqrt{a^2+x^2}}$
   (m) $\dfrac{1}{\sqrt{x^2-a^2}}$
   (n) $\dfrac{1}{x\sqrt{x^2-1}}$
   (o) $a^x, \; a > 0$
   (p) $\dfrac{1}{2-x^2}$
   (q) $\dfrac{1}{ax+b}$
   (r) $\dfrac{\sin(\tan^{-1} x)}{1+x^2}$
   (s) $\dfrac{1}{1+\tan x}$
   (t) $\dfrac{1}{a\sin x + b\cos x}$
   (u) $\sin^4 x$
   (v) $\sqrt{a^2-x^2}$
   (w) $\sqrt{a^2+x^2}$
   (x) $\sqrt{x^2-a^2}$
   (y) $\dfrac{1}{\sqrt{5x^2-2x}}$

2. Evaluate the following antiderivatives:

   (a) $\displaystyle\int \frac{x^2dx}{\sqrt{x^6+a^6}}$
   (b) $\displaystyle\int \frac{dx}{\sqrt{(x-a)(x-b)}}$
   (c) $\displaystyle\int \frac{dx}{(x+1)(x+2)}$
   (d) $\displaystyle\int \frac{3x-2}{(x+1)^2(x+3)}dx$
   (e) $\displaystyle\int \frac{x^2dx}{(x^2+1)(x^2+4)}$
   (f) $\displaystyle\int \frac{dx}{x^4-1}$
   (g) $\displaystyle\int \frac{dx}{x^4+1}$
   (h) $\displaystyle\int \frac{dx}{x(x^n+1)}$

3. Evaluate the following antiderivatives:

   (a) $\displaystyle\int e^{ax}\sin bx \; dx$
   (b) $\displaystyle\int e^{ax}x^4 \; dx$
   (c) $\displaystyle\int x\sec^2 x \; dx$
   (d) $\displaystyle\int \sec^3 x \; dx$
   (e) $\displaystyle\int (\sin^{-1} x)^2 \; dx$
   (f) $\displaystyle\int \frac{x\cos^{-1} x \; dx}{\sqrt{1-x^2}}$
   (g) $\displaystyle\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

4. Prove: $$\int_0^{\pi/2} \sin^m x\cos^n x dx = \begin{cases} \frac{(m-1)}{(m+n)}\frac{(m-3)}{(m+n-2)}\cdots\frac{2}{(n+2)} \times \frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\cdots\frac{1}{2}\times\frac{\pi}{2}, & \text{if } m \text{ and } n \text{ are even} \\ \frac{(m-1)}{(m+n)}\frac{(m-3)}{(m+n-2)}\cdots\frac{3}{(n+3)} \times \frac{1}{(n+1)}, & \text{if } m \text{ is odd} \\ \frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\cdots\frac{1}{2}\times\frac{\pi}{2}, & \text{if } m=0 \text{ and } n \text{ is even} \\ \frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\cdots\frac{2}{3}, & \text{if } m=0 \text{ and } n \text{ is odd} \end{cases}$$

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