Department of Mathematics

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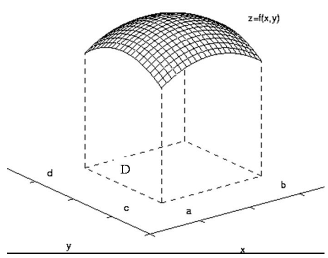
Notes on 'Real functions of several real variables'

Let $D \subset \mathbb{R}^2$ be open; we define a real function of two variables to be a map $f : D \to \mathbb{R}$ and we write $z = f(x, y),\ (x, y) \in D$. You can visualize the function $z = f(x, y)$ as a three dimensional graph as in the Fig. below where corresponding to each value of $(x, y) \in D$ the function value $f(x, y)$ along the $z-$axis is assigned to the function. Here the domain set D is the rectangle $[a, b] \times [c, d]$.



Example 1: $z = f(x, y) := x^2 + y^2$ where $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$. Here the domain of function f is the square $[-1, 1] \times [-1, 1]$.

Definition 1: A function $f : D \to \mathbb{R}$ of two variables is said to have partial derivative at a point $(a, b) \in D$ w.r.t. x if for every $\epsilon > 0$, there exist real numbers λ and $\delta > 0$ such that $|x - a| < \delta \ \Rightarrow \ \left|\dfrac{f(x, b) - f(a, b)}{x - a} - \lambda\right| < \epsilon$. If f has partial derivative w.r.t. x at point (a, b) we say that it is λ. We denote this partial derivative by $\dfrac{\partial f}{\partial x}(a, b) := \lambda$. Similarly we can define partial derivative of f w.r.t. y at point (a, b) to be $\dfrac{\partial f}{\partial y}$.

Definition 2: The function f of two variables is said to be differentiable at point $(a, b) \in D$ if for every $\epsilon > 0$ there is a $\delta > 0$ and a linear map $T_{(a,b)} : D \to \mathbb{R}$ such that $\sqrt{(x - a)^2 + (y - b)^2} < \delta \ \Rightarrow \ \left|\dfrac{f(x, y) - f(a, b) - T_{(a,b)}(x - a, y - b)}{\sqrt{(x - a)^2 + (y - b)^2}}\right| < \epsilon$. If f is differentiable at point (a, b) the linear map $T_{(a,b)}$ is called the derivative of f at point (a, b). Note that from above two definitions it follows that

$$T_{(a,b)} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \dfrac{\partial f}{\partial x}(a, b) & \dfrac{\partial f}{\partial y}(a, b) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where we have identified the point (x, y) by the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$. The underlying matrix of the linear map $T_{(a,b)}$ is called the Jacobain matrix of $T_{(a,b)}$.

Definition 3: A function $f(x, y)$ is said to be homogeneous of degree n if for every real number $\lambda \neq 0$ $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. Let $y \neq 0$ and f be differentiable function of x and y. Then we have the following

Theorem 1:(Euler's theorem on homogeneous functions)

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Definition 4: Let $z = f(x, y)$ be as above and let δz be small increment in z corresponding to small increments δx and δy in x and y respectively, s.t. $z + \delta z = f(x + \delta x, y + \delta y)$ and that $\delta z = f(x + \delta x, y + \delta y) - f(x, y)$. As $\delta x \to 0$ and $\delta y \to 0$, $\delta z \to 0$ and we obtain the following

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

dz as above is called as the total derivative (or differential 1-form) of f at point (x, y).

Discussion: To visualize total derivative dz of $z = f(x, y)$ you can consider two neighboring points $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ on the surface define by the function $F(x, y, z) := -z + f(x, y) = 0$ such that the vector $\overrightarrow{PQ} = (\delta x, \delta y, \delta z)$ becomes a tangent vector to the surface at point (x, y, z) as $Q \to P$ along some curve passing through the surface. If f possesses partial derivative w.r.t. x and y and z we have the total derivative $dF = \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy + \dfrac{\partial F}{\partial z}dz = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy - dz = 0$. This means $\lim_{Q \to P} \nabla F \cdot \overrightarrow{PQ} = 0$ where $\nabla F = \left(\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}\right)$; we see that $\nabla F \perp \overrightarrow{PQ}$ as $Q \to P$. We conclude that ∇F is a normal to the surface defined by the equation $F(x, y, z) = 0$ at point (x, y, z).

Definition 5: An equation of the form $F(x, y) = 0$ is called an implicit equation in x and y. y is said to be an implicit function of x if there exists an implicit equation $F(x, y) = 0$ which can be solved for y as a function of x in some open subset of the domain of F. e.g. $F(x, y) = x^2 + y^2 - 1 = 0$ is an implicit equation in x and y where we can solve $y = \sqrt{1 - x^2}$ as an implicit function of x in a neighborhood of the point $(0, 1)$.

Application: Let y be an implicit function of x such that $F(x, y) = 0$. Then total derivative $dF = 0$ i.e. $\dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy = 0$ which give s us $\dfrac{dy}{dx} = -\dfrac{\partial F}{\partial x}\Big/\dfrac{\partial F}{\partial y}$ provided that the denominator here should not vanish.

Theorem 2(Chain Rule): Let $z = f(x, y)$ be smooth and $x = x(u, v)$ and $y = y(u, v)$ be smooth functions of u and v such that the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)} \neq 0$ on the domain of x, y as well as f. Then the following holds:

$$\begin{pmatrix} \dfrac{\partial z}{\partial u} \\ \dfrac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \dfrac{\partial z}{\partial x} \\ \dfrac{\partial z}{\partial y} \end{pmatrix} \tag{0.1}$$

The matrix $\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{pmatrix}$ is called as the Jacobian matrix of transformation $(x, y) \to (u, v)$ of coordinates.

Example: Consider the transformation of coordinates $(x, y) \to (r, \theta)$ s.t. $x = r\cos\theta,\ y = r\sin\theta$; Then using the chain rule we have $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = \cos\theta\dfrac{\partial z}{\partial x} + \sin\theta\dfrac{\partial z}{\partial y}$. Similarly $\dfrac{\partial z}{\partial \theta} = -r\sin\theta\dfrac{\partial z}{\partial x} + r\cos\theta\dfrac{\partial z}{\partial y}$. Note that here the Jacobian of transformation is the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$ with determinant r.

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Assignment #6:Real functions of several real variables

1. Obtain the partial derivatives of the following functions:

   (a) $f(x, y) := \sin xy$ at point $(\pi, 1/2)$

   (b) $f(x, y) := x^2 + y$ at $(1, 2)$

   (c) $f(x, y) : \dfrac{x^2 - y^2}{x^2 + y^2}$ at $(1, 0)$

   (d) $f(x, y) := (y^2 - 2)e^{x+y}$ at $(-1, 1)$

2. Verify Euler's theorem for the following functions:

   (a) $f(x, y) = \tan\left(\dfrac{x^3 - y^3}{x^3 + y^3}\right)$

   (b) $f(x, y) = x - y + y^2/x, \ x \neq 0$

   (c) $f(x, y) = \dfrac{x}{y} + \dfrac{y}{x}, \ x, y \neq 0$

   (d) $f(x, y) = x^n \varphi\left(\dfrac{x}{y}\right), \ y \neq 0.$

3. Let $u := \sin^{-1}\left(\dfrac{x^5 - y^5}{x^2 + y^2}\right)$. Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$.

4. Determine the total derivative of the following functions:

   (a) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

   (b) $f(x, y) = x - y + xy^2 + e^{x-y}$

   (c) $f(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^{5/2}}$

   (d) $f(x, y, z) = \dfrac{\sin x}{x + y + z}$

5. Using the chain-rule and transformation of coordinates obtain the Laplace equation $\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2} = 0$ in cylindrical polar coordinates.

6. Derive the equations of tangent plane and tangent normal line to the surface of the cone $x^2 + y^2 = 2z^2$ at point $(1, 1, 1)$?

7. Obtain equation of a surface satisfying the relation $xdx + ydy - 2zdz = 0$.

8. Obtain $\dfrac{dy}{dx}$ from the implicit equation $x^2 - y^2 = 2 - x$ using partial derivatives.

9. Let $f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } x = 0 \end{cases}$ Evaluate $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$ at point $(0, 0)$.

10* Prove that for two real variables x and y and the expression $M(x, y)dx + N(x, y)dy$ where M and N are smooth functions of x, y, there exists a smooth function $F(x, y)$ such that $dF = M(x, y)dx + N(x, y)dy$ if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ for all (x, y) in the interior of the domain of definition of M and N.

11* Prove that a solution curve satisfying $dF = Mdx + Ndy = 0$ is given by

$$\int_{y\text{-constant}} Mdx + \int (\text{terms of } \ N \text{ not containing } x)dy = \text{constant}$$

12* If there exists a smooth function $F(x, y)$ such that $dF = M(x, y)dx + N(x, y)dy$ and $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \varphi(x)$ then prove that $\mu(x, y) = \exp\left\{\int \varphi(x)dx\right\}$.

13* If there exists a smooth function $F(x, y)$ such that $dF = M(x, y)dx + N(x, y)dy$ and $\dfrac{-1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \varphi(y)$ then prove that $\mu(x, y) = \exp\left\{\int \varphi(y)dy\right\}$.

14* Solve the following differential forms:(a) $xdy - ydx = 0$ (b) $x^2dy - y^2dx = 0$ (c) $\dfrac{dy}{dx} = \dfrac{1 + x^2y^3}{-x^3y^2}$

[2]The problems marked with an asterisk * will be revisited while studying exact differential equations