Department of Mathematics

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Assignment #5:Applications of Derivatives

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In this assignment, we will discuss some applications of derivatives of "smooth" functions of the form $f : [a, b] \to \mathbb{R}$. Recall Lagrange's mean value theorem: *If $f : [a, b] \to \mathbb{R}$ be a function such that (a) f is continuous on closed interval $[a, b]$ (b) f is differentiable in open interval (a, b), then there is a $c \in (a, b)$ for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ or $f(b) - f(a) = (b - a)f'(c)$.*

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be increasing on an interval $[a, b]$ if for all $x_1, x_2 \in [a, b]$ $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$. f is said to be *strictly* increasing on $[a, b]$ if for all $x_1, x_2 \in [a, b]$ $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Definition 2: The function f is said to be decreasing on an interval $[a, b]$ if for all $x_1, x_2 \in [a, b]$ $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$. f is said to be *strictly* decreasing on $[a, b]$ if for all $x_1, x_2 \in [a, b]$ $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Remark: We write $f \uparrow$ and $f \downarrow$ respectively for increasing and decreasing functions. If f is neither increasing nor decreasing or equivalently it is both increasing and decreasing on an interval we write $f \uparrow\downarrow$. Note that if $f \uparrow\downarrow$ over an interval then f is a constant function.

Definition 3: The function f is said to have a local maximum value at a point $x \in (a, b)$ if there exist a $\delta > 0$ such that $|t - x| < \delta \Rightarrow f(t) \leq f(x)$. f is said to have local minimum value at point $y \in (a, b)$ if there is a $\eta > 0$ such that $|s - y| < \eta \Rightarrow f(s) \geq f(y)$.

Remark: If f has local maximum at point $x \in (a, b)$ then by definition, there is a $\delta > 0$ such that $|t - x| < \delta \Rightarrow f(t) \leq f(x)$. Then $x - \delta < t < x \Rightarrow f(t) \leq f(x)$. In particular when f is differentiable at x we may choose a $\delta^* < \delta$ such that $f \uparrow$ on $(x - \delta^*, x]$ and $f \downarrow$ on $[x, x + \delta^*)$. If we consider $t \in (x - \delta^*, x)$ where $f \uparrow$ the expression $\dfrac{f(x) - f(t)}{x - t} \geq 0$ while for $t \in (x, x+\delta^*)$ $\dfrac{f(x) - f(t)}{x - t} \leq 0$. Therefore $f'(x) = \lim_{t \to x} \dfrac{f(x) - f(t)}{x - t} \lesseqgtr 0 = 0$ since $f'(x)$ exists. Similarly if f has a local minimum at some $y \in (a, b)$ then $f'(y) = 0$.

Now assuming f to be smooth over (a, b) then for every $x_1, x_2 \in (a, b)$ apply Lagrange's theorem on interval $[x_1, x_2]$ to find a $x \in (x_1, x_2)$ such that

$$f(x_1) - f(x_2) = (x_1 - x_2)f'(x) \begin{cases} > 0 & \text{iff } f'(x) < 0 \\ < 0 & \text{iff } f'(x) > 0 \end{cases} \qquad (0.1)$$

Then the Eq.(0.1) holds for all y_1, y_2 in all subintervals $(x - \eta, x + \eta) \subset (x_1, x_2)$, $\eta > 0$. Consequently, it follows that $f \uparrow$ at x iff $f'(x) > 0$ and $f \downarrow$ at x iff $f'(x) < 0$.

Second Derivative Test: Let f be as above and three times continuously differentiable on (a, b). Using Taylor's theorem, for all $(x, x + h) \subset (a, b)$ there is a $c \in (x, x + h)$ such that $f(x+h) - f(x) = hf'(x) + \dfrac{h^2}{2}f''(x) + \dfrac{h^3}{3!}f'(c)$. Let f be a point of local extreme value, then $f'(x) = 0$. Suppose if $f''(x) \neq 0$ then as $f'(c)$ is finite, we can choose h sufficiently small such that $\text{sign}\{f(x + h) - f(x)\} = \text{sign}\{f''(x)\} = \text{sign}\{h^3 f'(c)\}$. This means $f''(x) < 0 \iff f(t) \leq f(x)$ for all $t \in (x, x + h)$. Similarly applying the above analysis on sub interval $(x - h, x)$ we find that $f''(x) < 0 \iff f(t) \leq f(x)$ for all $t \in (x - h, x)$. Combining the above two results establish that $f''(x) > 0 \iff f$ has local maximum at x. With the similar argument it follows that $f''(x) < 0 \iff f$ has local minimum at x. Note that the above test fails for if $f''(x) = 0$ then we need to look for an analogous third derivative test and so on!

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1. Determine the intervals where the following functions are (i) $\uparrow$ (ii) $\downarrow$. Also find the points of local extreme values for each of these functions.

   (a) $f(x) := \cos x$ on $[0, 2\pi]$

   (b) $f(x) := x^2 + x$ on $(-\infty, \infty)$

   (c) $f : \mathbb{R} \to \mathbb{R}$ where $f'(x) := (x^2 - 5)(x^2 - 4)(x^2 - 3)$

   (d) $f(x) := a \sin x + b \cos x$ on $[-\pi, \pi]$, $a, b \in \mathbb{R}$

   (e) $f(x) := \dfrac{x^2 - 1}{x^2 + 1}$ on $[-1, 1]$

   (f) $\sin^{-1} : [-1, 1] \to [3\pi/2, 2\pi]$

   (g) $f(x) = a^x,\ a > 0$ on $\mathbb{R}$

   (h) $f(x) = -2x^3 - 9x^2 - 12x + 1$ on $(-\infty, \infty)$

2. Determine set of the point of local maxima and local minima for the following functions. Also determine the local maximum and local minimum values.

   (a) $f(x) = \tan x$ on $[0, \pi] - \{\pi/2\}$

   (b) $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$ on $(-\infty, \infty)$

   (c) $f(x) = 2x^3 - 6x^2 + 6x + 5$ on $(-\infty, \infty)$

   (d) $f(x) = 2x^3 - 15x^2 + 36x + 1$ on $(-\infty, \infty)$

   (e) $f(x) = \sin^{-1} x$ on $[-1, 1]$

   (f) $f(x) = 2 \sin x + 3 \cos x$, on $(0, 2\pi)$

3. Find two positive integers whose sum is 15 and the sum of whose square is minimum.

4. Find shortest distance of the point $(0, c)$ from the parabola $y = x^2$. [Shortest distance of a point $(a, b) \in \mathbb{R}^2$ from a planer cartesian curve C defined by $f(x, y) = 0$ is defined as $d = \min_{(x,y) \in C}\{\sqrt{(x - a)^2 + (y - b)^2}\}$]

5. Prove that among all rectangles of a given fixed perimeter P, square encloses maximum area.

6. An open topped box is to be constructed by removing equal square pieces from each corned of a 3 meters by 8 meters rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

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