Department of Mathematics

Guru Nanak Dev University, Amritsar-143005

Assignment #4: Mean value theorems

August 8, 2011

Theorem 0(Extreme value theorem): If $f : [a, b] \to \mathbb{R}$ is continuous then there exist two real numbers $\alpha,\ \beta \in [a, b]$ such that $f(\alpha) \le f(x) \le f(\beta)$ for all $x \in [a, b]$.

Theorem 1(Rolle's Theorem): If $f : [a, b] \to \mathbb{R}$ be a function such that

(a) f is continuous on closed interval $[a, b]$

(b) f is differentiable in open interval (a, b)

(c) $f(a) = f(b)$

then there is a $c \in (a, b)$ for which $f'(c) = 0$.

Proof: If f is a constant function, then we are done so let f is not a constant function. Since f is continuous on $[a, b]$ by theorem 0 there is an $\alpha \in (a, b)$ such that $f(x) \le f(\alpha)$ for all x. Note that $\alpha \ne a$ or b because $f(a) = f(b)$ and f is not constant function. For a small $\delta > 0$ consider the set of all x for which $|\alpha - x| < \delta$. Then the quantity

$$\frac{f(x) - f(\alpha)}{x - \alpha} \begin{cases} > 0 & \text{if } (\alpha - \delta) < x < \alpha \\ < 0 & \text{if } \alpha < x < (\alpha + \delta) \end{cases}$$

$$\Rightarrow f'(\alpha) = \lim_{x \to \alpha} \left(\frac{f(x) - f(\alpha)}{x - \alpha} \right) = \begin{cases} \text{LHL} \ge 0 & \text{if } x < \alpha \\ \text{RHL} \le 0 & \text{if } \alpha < x \end{cases}$$

Since $f'(\alpha)$ exists as $\alpha \in (a, b)$, therefore LHL=RHL above, which is possible only if $f'(\alpha) = 0$.

Theorem 2(Lagrange's Mean Value Theorem): If $f : [a, b] \to \mathbb{R}$ be a function such that

(a) f is continuous on closed interval $[a, b]$

(b) f is differentiable in open interval (a, b)

then there is a $c \in (a, b)$ for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof: Let $\varphi(x) := f(x)(b - a) - \{f(b) - f(a)\}x$. Then clearly φ satisfies the first two axioms of the Rolle's theorem since these are satisfied by f and the function $I(x) = x$. We only need to verify the third axiom and for this we see that, $\varphi(a) - \varphi(b) = (b - a)\{f(a) - f(b)\} - \{f(b) - f(a)\}(a - b) = 0$ or $\varphi(a) = \varphi(b)$. Therefore by Rolle's theorem, there exist $c \in (a, b)$ for which $\varphi'(c) = 0$ or $f'(c)(b - a) - \{f(b) - f(a)\}c = 0$ from which the result follows.

Theorem 3(Cauchy's Theorem): If $f, g : [a, b] \to \mathbb{R}$ be two functions such that

(a) f and g are continuous on closed interval $[a, b]$

(b) f and g are differentiable in open interval (a, b)

then there is a $c \in (a, b)$ for which $f'(c)\{g(b) - g(a)\} - \{f(b) - f(a)\}g'(c) = 0$. This theorem reduces to Lagrange's theorem if we take $g(x) = x$.

Proof: Define $\varphi(t) = f(x)\{g(b) - g(a)\} - \{f(b) - f(a)\}g(x)$ on $[a, b]$ and apply Rolle's theorem to conclude.

Theorem 4(Taylor's Theorem): Let $f : [a, b] \to \mathbb{R}$ be *s.t.*

(a) f is continuous on closed interval $[a, b]$

(a) $f, f', f'', ..., f^n$ exist and are continuous on (a, b), $n \in \mathbb{Z}^+$

then there is a $c \in (x, x + h) \subset (a, b)$, $h > 0$ *s.t.* for any $x \in [a, a + h) \subset [a, b)$ the following holds

$$\boxed{f(x + h) = f(x) + \frac{h}{1!} f'(x) + \cdots + \frac{h^{n-1}}{(n - 1)!} f^{n-1}(x) + \frac{h^n}{n!} f^n(c).}$$

Proof: Let $f(x+h) := f(x) + \dfrac{h}{1!} f'(x) + \cdots + \dfrac{h^{n-1}}{(n - 1)!} f^{n-1}(x) + \dfrac{h^n}{n!}\xi$ for some real number ξ. Define $\varphi(t) := -f(x + h) + f(t) + \dfrac{x + h - t}{1!} f'(t) + \cdots + \dfrac{(x + h - t)^{n-1}}{(n - 1)!} f^{n-1}(t) + \dfrac{(x + h - t)^n}{n!}\xi$. Then clearly $\varphi(x + h) = \varphi(x) = 0$ and φ also satisfies the other two axioms of the Rolle's theorem since these are satisfied by $f, f', ..., f^{n-1}$. Therefore by Rolle's theorem, there is a $c \in (x, x + h)$ for which $\varphi'(c) = 0$ which gives us $\xi = f^n(c)$.

---

1. Verify Rolle's theorem for the following functions:

   (a) $f(x) := \sin x$ on $[0, 2\pi]$

   (b) $f(x) := x^2 + x$ on $[-1, 0]$

   (c) $f(x) := a \sin x + b \cos x$ on $[-\pi, \pi]$, $a, b \in \mathbb{R}$

   (d) $f(x) := \dfrac{x^2 - 1}{x^2 + 1}$ on $[-1, 1]$

2. Verify Lagrange's mean value theorem for the following functions:

   (a) $f(x) = \tan x$ on $[0, \pi/4]$

   (b) $f(x) = \dfrac{b}{a}\sqrt{a^2 - x^2}$ on $[-a/2, a]$, $0 < b < a$

   (c) $f(x) = \cos 3x$ on $[0, \pi/3]$

   (d) $f(x) = e^{2x+3}$ on $[-4, 2]$

   (e) $f(x) = \log x$ on $[1, 5]$.

   (f) $f(x) = 2^x$ on $[0, 1]$

3. Using Taylor's theorem write possible finite order expansions for the following functions about the point $x = 0$:

   (a) $f(x) = \sin x$ on $[0, \pi/2]$

   (b) $f(x) = |x|^5 - x^5$ on $[-1, 1]$

   (c) $f(x) = \tan x$ on $[0, \pi/4]$

   (d) $f(x) = \tan^{-1} x$ on $(-\infty, \infty)$

   (e) $f(x) = \sin^{-1} x$ on $[-1, 1]$

   (f) $f(x) = \log(1 + x)$ for $|x| < 1$

4. Express the function $f(x) := x^5 - 2x^4 + x^3 - 2$ in powers of $x - 1$ using Taylor's theorem.

5. If $|x| < 10^{-1}$ what is the maximum error in calculation of $\sin x$ using Taylor's theorem about the point $x = 0$ for $n = 3$?

6. Approximate $\sin 1^\circ$ using (a) Lagrange's mean value theorem (b) Taylor's theorem.

7. Calculate value of π correct to 8 decimal places. [**Hint**: Use Taylor's theorem for function $f(x) = \tan^{-1} x$ & utilize the identity $\tan^{-1} 1 = \frac{\pi}{4}$. Choose suitable n to obtain the desired accuracy]

---

[1]E-mail: sonumaths@gmail.com