Department of Mathematics

Guru Nanak Dev University, Amritsar-143005
Assignment #3: B.Sc.(Hons) Physics Semester I
MTL 131: Derivatives
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In some of the following problems, you should use the chain rule to obtain the derivative.

Theorem 1(Chain Rule): If $f : A \to f(A)$ and $g : f(A) \to C$ are differentiable at $x = a$ and $f(x) = f(a)$ respectively, then $g \circ f : A \to C$ is differentiable at $x = a$ and

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

Example 1: If you have say $f(x) := \cos\{\sqrt{x+1}\}$ and you are asked to find $f'(x)$ then you may define two functions function $F(x) := \sqrt{x+1}$ with $F'(x) = \frac{1}{2\sqrt{x+1}}$, $x > -1$ and $G(y) = \cos y$ with $G'(y) = -\sin(y)$ so that now using chain rule we have the following:

$$f'(x) = (G \circ F)'(x) = G'(F(x)) \times F'(x) = -\sin(F(x)) \times F'(x) = -\sin\{\sqrt{x+1}\} \times \frac{1}{2\sqrt{x+1}}$$

Example 2: Some questions are based on other techniques of obtaining derivatives like substitution or taking logarithm etc. e.g. If you are asked to obtain derivative of $y = \cos^{-1}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)$, $a > 0$, $|x| < a$, try putting $x = a\tan\theta$ s.t. $y = \cos^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) = \cos^{-1}(\cos\theta/2) = \theta/2$. Consequently $y' = (y \circ \theta)'(x) = y'(\theta) \times \theta'(x) = \frac{1}{2} \times \frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{2(a^2 + x^2)}$.

Example 3: If $y = x^x, x > 0$ we should proceed for derivative after taking logarithm i.e. $\log y = x \log x$ so that

$$y'/y = 1 \times \log x + x/x = 1 + \log x \quad \text{or} \quad y' = y(1 + \log x) = x^x(1 + \log x).$$

Notation: Let $y_0 = f(x)$, we will denote the successive derivatives $f'(x), f''(x), ...$ by $y_1, y_2, ...$ i.e. $y_k = f^{<k>}(x)$=$k-$th derivative of $f(x)$. The last question below is based on the Leibnitz product rule:

Theorem 2(Leibnitz Rule): Let $u(x)$ and $v(x)$ be $n-$times differentiable functions of $x \in \mathbb{R}$, then the product $u(x)v(x)$ is differentiable at x and

$$(uv)_n = n_{C_0}u_n v_0 + n_{C_1}u_{n-1}v_1 + \cdots + n_{C_n}u_0 v_n, \quad n_{C_r} := \frac{n!}{r!(n-r!)}.$$

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1. Determine the derivative of following functions at point x:

   (a) $f(x) := \sin\{x^2 + 5\}$

   (b) $f(x) := \frac{\sin\{ax + b\}}{\cos\{cx + d\}}$

   (c) $f(x) := \cos\{x^2\}\sin^2\{x^5\}$

   (d) $y := \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, $|x| < \frac{1}{\sqrt{3}}$

   (e) $y := \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$, $|x| < \frac{1}{\sqrt{2}}$

   (f) $y := \sqrt{e^{\sqrt{1+x}}}, x > -1$

   (g) $y := \cos x \cos 2x \cos 3x$

   (h) $y := (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$

   (i) $y := \left(x + \frac{1}{x}\right)^x + x^{\left(x + \frac{1}{x}\right)}$

   (j) $y := \sqrt{\frac{(x - 3)(x^2 + 4)}{3x^2 + 4x + 5}}$

2. Determine the derivative $\frac{dy}{dx}$ from the following:

   (a) $x := 2t;\ y := 2t^4$

   (b) $x := a(\theta - \sin\theta); y := a(1 + \cos\theta)$

   (c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} - 3xy = 0$

   (d) $\sin(x + y) = 1$

   (e) $x^2 + y^2 + 2gx + 2fy + c = 0$ for f, g, c constants.

   (f) $x = \sin t;\ y := \cos t$

3. Obtain $n-$th derivative of the following for given n positive integer or otherwise:

   (a) $y = \frac{x}{1 + x}$

   (b) $y = x^3 + \tan x$ for $n = 2$

   (c) $y = A\sin(2x+3) + B\cos(ax+b)$

   (d) $y = x^n e^{ax}$

   (e) $y = x^2\sin(ax + b)e^{cx}$

   (f) $y = \sin^{-1}x$ for $n = 2$

4. (a) If $y := (\tan^{-1}x)^2$ show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$.

   (b) If $y := c_1 e^{ax} + C_2 e^{bx}$ show that $\frac{d^2y}{dx^2} - (a + b)\frac{dy}{dx} + aby = 0$.

5. If $y := \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell & m & n \\ a & b & c \end{vmatrix}$ prove that $\frac{dy}{dx} := \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell & m & n \\ a & b & c \end{vmatrix}$