Department of Mathematics

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Assignment #2: B.Sc.(Hons) Physics Semester I

MTL 131: Continuity, Differentiability

July 28, 2011

Definition 1: $f(x)$ is said to be continuous at a point $x = a$ of its domain if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon$$

Definition 2: $f(x)$ is said to be differentiable at a point $x = a$ of its domain with derivative $f'(a) \in \mathbb{R}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon$$

Remark 1: If f is continuous at $x = a$, we write $\lim_{x \to a} f(x) = f(a)$.

Remark 2: If f is differentiable at $x = a$ with derivative $f'(a)$, we write $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$.

Theorem 1: If a function f is differentiable at a point $x = a$ with derivative $f'(a)$ then f is continuous at $x = a$.

Proof: Let f is differentiable at a point $x = a$ with derivative $f'(a)$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - a| < \delta_1 \quad \Rightarrow \quad \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon$$

As

$$\left| \frac{f(x) - f(a)}{x - a} \right| - |f'(a)| \le \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon,$$

we need to find a $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| \le (\epsilon + |f'(a)|)|x - a| < \epsilon.$$

This can be done by defining $\delta := \min \left\{ \delta_1, \frac{\epsilon}{\epsilon + |f'(a)|} \right\}$. Result follows now.

Remark 2: Converse of the theorem 1 is not true as there are continuous functions which are not differentiable. An example is $f(x) = |x|$ which is continuous at $x = 0$ but fails to have derivative at $x = 0$ as can be seen readily from the following: $\lim_{x \to 0} \frac{|x| - 0}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$ which does not exist!

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1. Check the continuity of the following functions:

   (a) $f(x) := x - |x|$ at point $x = 2$.

   (b) $f(x) := \frac{|x|}{x}$ at $x = 0$.

   (c) $f(x) := \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$.

   (d) $f(x) := \begin{cases} x - 1 & \text{if } x > 1 \\ 2 + x & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$ at $x = 1$.

   (e) $f(x) := \begin{cases} \frac{x^2 - 4x + 4}{x^2 - 4} & \text{if } x > 2 \\ (x - 2)^2 & \text{if } x < 2 \\ 0 & \text{if } x = 2 \end{cases}$, at $x = 2$.

   (f) $f(x) := \begin{cases} \frac{\sin 2x}{x} & \text{if } x > 0 \\ 2\frac{e^x - 1}{x} & \text{if } x \le 0 \end{cases}$, at point $x = 0$.

   (g) $f(x) := \begin{cases} \frac{\sin 4x}{\sin 3x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$, at point $x = 0$.

2. If the function $f(x) = \begin{cases} -a + bx & \text{if } x < 1 \\ 8 & \text{if } x = 1 \\ b - ax & \text{if } x > 1 \end{cases}$; is continuous at $x = 1$ what are the possible values of a and b?

3. Let f and g be two real functions. If f and g are continuous at a point $x = a$, then prove that so are $f(x) \pm g(x)$ and $f(x)g(x)$. Hence conclude that the functions $x + \sin(2x)$ and $x \sin(ax)$ are continuous at all $x \in \mathbb{R}$.

4. If $f(x) := x + 2$ and $g(x) := 2 - x^2$, evaluate $(f \circ g)'(1)$ and $(g \circ f)'(1)$.

5. Check whether the function $f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable at $x = 0$?

6. Check differentiability of the function $f(x) := |x - 2|^3$ at $x = 2$.