Department of Mathematics

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Assignment #1: B.Sc.(Hons) Physics Semester I

MTL 131: Function Limit

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Definition 1(Limit): A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a limit ℓ at a point $x = a,\ a \in \mathbb{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - \ell| < \epsilon$$

Note that in this definition, $\delta = \delta(\epsilon)$ i.e. δ is a function of ϵ. We may define left hand limit (LHL) and right hand limit (RHL)of f at point $x = a$ to be as follows:

Definition 2(LHL): $f(x)$ is said to have left hand limit ℓ_1 at point $x = a$ if for every $\epsilon > 0$, there is a $\delta_1 > 0$ such that

$$a - x < \delta_1 \quad \Rightarrow \quad |f(x) - \ell_1| < \epsilon$$

Definition 3(RHL): $f(x)$ is said to have right hand limit ℓ_2 at point $x = a$ if for every $\epsilon > 0$, there is a $\delta_2 > 0$ such that

$$x - a < \delta_2 \quad \Rightarrow \quad |f(x) - \ell_2| < \epsilon$$

Remark 1: If left hand Limit of a function f at a point $x = a$ is ℓ_1 we write $\lim_{x \to a^-} f(x) = \ell_1$.

Remark 2:. If right hand Limit of a function f at a point $x = a$ is ℓ_2 we write $\lim_{x \to a^+} f(x) = \ell_2$.

Remark 3:. If Limit of a function f at a point $x = a$ is ℓ we write $\lim_{x \to a} f(x) = \ell$.

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1. Check the correctness of the following identities via definition of limit:

   (a) $\lim_{x \to 0} \sqrt{1 + x + x^2} - x = 1$

   (b) $\lim_{x \to 2} x - |x - 2| = 2$

   (c) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = \frac{1}{10}$

   (d) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a,\ a > 0$

   (e) $\lim_{x \to a} x^n = a^n,\ n \in \mathbb{Z}^+$

   (f) $\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 - 4} = 0$

   (g) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

   (h) $\lim_{x \to 0} (1 - \sin x + |x|) = 1$

   (i) $\lim_{x \to 2} \frac{x^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x - 2} = \frac{2^{-\frac{2}{3}}}{3}$

   (j) $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \frac{1}{2}$

   (k) $\lim_{x \to 0} \frac{\sin 4x}{\sin 3x} = \frac{4}{3}$

2. Evaluate $\lim_{x \to \infty} \frac{\sqrt{2x^2 + x - 1}}{\sin x} e^{-x}$

3. Determine α if $\lim_{x \to 0} \frac{1 + x \sin \alpha x - \cos x}{x^2} = 0$.

4. If $f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, check $\lim_{x \to 0} f(x)$.

5. Let $f(x) = \begin{cases} a + bx & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ b - ax & \text{if } x > 1 \end{cases}$; if $\lim_{x \to 1} f(x) = f(1)$ what are the possible values of a and b?

6. Prove that the area A_n of a regular polygon with $n-$sides and circum-radius r is $A_n := n\frac{r^2}{2} \sin\left(\frac{2\pi}{n}\right)$. Show that $\lim_{n \to \infty} A_n = \pi r^2$.

7. Let f and g be two real functions. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, prove that $\lim_{x \to a} f(x) \pm g(x)$ and $\lim_{x \to a} f(x)g(x)$ also exist. Hence show that $\lim_{x \to 0} \left(\frac{\sin x}{x} + x^3 - 1\right)$ and $\lim_{x \to 0} \frac{\sin 2x}{x^3 - x}$ exist.

8. Let $f(x) = \cos(2x + 3)$. Evaluate $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$.

9. Evaluate $\lim_{x \to \frac{\pi}{2}} (\cos x)^{\cos x}$ [**Hint:** Let $y = (\cos x)^{\cos x}$ then $\lim_{x \to \frac{\pi}{2}} \log y = \lim_{x \to \frac{\pi}{2}} \frac{\log \cos x}{\sec x}$. Now solve using L'Hospital rule.]

10. If $f(x) := x - |x|$ and $g(x) := 2 - |x|$ Evaluate $\lim_{x \to 2} (f \circ g)(x)$ and $\lim_{x \to 2} (g \circ f)(x)$.