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Notes on 'Linear Ordinary Differential Equations'

a couple of definitions.

Definition 1:(Linear map) A map $f : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m,\ m,n \in \mathbb{Z}^+$ is called a linear map if for all $x, y \in A$ and $\alpha \in \mathbb{R}$, (a) $f(x+y) = f(x) + f(y)$ and (b) $f(\alpha x) = \alpha f(x)$.

Example 1: The derivative map $\frac{d}{dx} : A \to B$ such that A = set of differentiable real functions of $x \in \mathbb{R}$, B=set of derivatives of members of A w.r.t. x, is a linear map because: $\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g,\ \frac{d}{dx}(\alpha f) = \alpha \frac{d}{dx} f,\ \forall \alpha \in \mathbb{R}$.

Remark 1: If f and g are linear maps over same domain and codomain then their sum $f+g$ defined by $(f+g)(x) = f(x)+g(x)$ is also a linear map. Also for a constant c the map cf defined by $(cf)(x) = cf(x)$ is also a linear map if f is a linear map.

Remark 2: It follows that if $c_1, c_2, ..., c_n$ be scalars and $\varphi_1, \varphi_2, ..., \varphi_n$ be n linear maps over same domain and codomain, then their linear combination defined by $\varphi := c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n$ is also a linear map.

Definition 3:(Kernel) Let $L : A \to B$ be a surjective linear map from a vector space A to a vector space B over $\mathbb{R}$. We define kernel of L to be the subspace $\ker L := \{x \in A \mid L(x) = \mathbf{0}_B\}$.

Theorem 1: *Let $L : A \to B$ (A,B as above) be a surjective linear map. Then $\tilde{L} : \tilde{A} \to B$ defined by $\tilde{L}(\tilde{x}) := L(x)$ is invertible where $\tilde{x} := \{x+y \mid y \in \ker\{L\}\}$.*

Proof: We just need to prove that $\tilde{L}$ is 1-1. Let $\tilde{L}(\tilde{x}) = \tilde{L}(\tilde{y})$, it follows that $L(x-y) = 0$ which means $(x-y) \in \ker L \iff x + \ker L = y + \ker L$ i.e. $\tilde{x} = \tilde{y}$. Hence $\tilde{L}$ 1-1. Since it is already onto as so is L, it follows that $\tilde{L}$ is invertible.

Definition 2:(Derivative) A map $f : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at a point x if there is a linear map $Df(x) : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h \to 0} \left\{ \frac{f(x+h) - f(x) - Df(x)(h)}{h} \right\} = 0.$$

If f is differentiable at x then the map $Df(x)$ is called derivative of f at the point x. We write $Df(x)(h) = f'(x)h = Df(x)h$ where $Df(x)$ is the matrix of the linear map $Df(x)$.

Remark 3: Note that $\ker Df(x) := \{y \in \mathbb{R} \mid Df(x)(y) = 0\} = \{0\}$ provided $Df(x) \neq 0$. Then by theorem 1, the map $\widetilde{Df(x)}$ is invertible with inverse $\widetilde{Df(x)}^{-1} : \tilde{\mathbb{R}} \cong \mathbb{R} \to \mathbb{R}$ s.t.

$$\widetilde{Df(x)}(\tilde{h}) := Df(x)(h)$$

for every $f : \mathbb{R} \to \mathbb{R}$ which possesses an antiderivative. It follows that $Df(x)$ is also invertible and $Df(x)^{-1} = \int f(x)dx$. We will use the notation

$$D^{-1}(f(x)) \equiv Df(x)^{-1};\ D(f(x)) \equiv Df(x)$$

where $D : \mathcal{H} \to \mathcal{H}$ is a linear map from the functional space $\mathcal{H}$ of differentiable functions on $\mathbb{R}$ to itself sending every point $f(x)$ to its derivative $Df(x)$. Similarly $D^{-1} : \mathcal{H} \to \mathcal{H}$ is the inverse of the map D.

Remark 4: Let us write $D^n \equiv \frac{d^n}{dx^n}\ \forall n = 0, 1, 2...$ with $D^0 = I$ is the identity map. Then $D^n = (D \circ D \circ \cdots \circ D)(n-\text{times})$ is also a linear map. Therefore the map

$$F(D) := a_0 I + a_1 D + \cdots + a_n D^n,\ a_n \neq 0,\ a_i \in \mathbb{R}$$

is also a linear map.

Proposition 1: The linear map $(D + aI)f(x) : \mathbb{R} \to \mathbb{R}, a \neq 0$ is invertible for every function $f : \mathbb{R} \to \mathbb{R}$ having antiderivative s.t. $(D+aI)^{-1}(f(x)) = e^{-ax}D^{-1}e^{ax}(f(x))$.

Proof: $(D+aI)(e^{-ax}D^{-1}(e^{ax}f(x))) = e^{-ax}D(D^{-1}(e^{ax}f(x))) = f(x) = e^{-ax}D^{-1}D(e^{ax}f(x)) = e^{-ax}D^{-1}(e^{ax}(D+aI)(f(x)))$

Remark 5: For a real-analytic function f, the exponential operator defined by $e^{hD}(f(x)) = \sum_{n=0}^{\infty} \frac{h^n D^n(f(x))}{n!}$ is also invertible and defines the Taylor-series expansion of f. More precisely $e^{\pm hD}(f(x)) = f(x \pm h)$ so that $\left(e^{hD}\right)^{-1} = e^{-hD}$.

Definition 4:(Vector Field) A map $\mathbf{v} : S \subset \mathbb{R}^n \to \mathbb{R}^n$ associating every point $x \in S$ to a unique vector $\mathbf{v}(x)$ is called a vector field on $\mathbb{R}^n$, s.t. for any map $f : S \to S$ the following hold:

$$(f\mathbf{v})(x) = f(x)\mathbf{v}(x);\ (\mathbf{v} + \mathbf{w})(x) = \mathbf{v}(x) + \mathbf{w}(x)$$

where $\mathbf{w}$ is another vector field on S.

Definition 5:(Differential Equation) A differential equation on $S \subset \mathbb{R}^n$ is an equation of the form $\frac{dx}{dt} := \mathbf{v}(x)$. This definition is precise in a sense that there are equations involving derivatives but are not actually differential equations. For example the Eq. $\frac{dx}{dt} = x(x(t))$ is not a differential equation! *If you do not feel comfortable you can safely skip this definition. Since we will be studying only linear ordinary differential equations, let us define these now.*

Definition 6: (Linear ODE of order n) Let n be a positive integer. A linear ordinary differential equation (ODE) of order n on a subset of $\mathbb{R}$ is defined as

$$F(D)y = b(x)$$

where $F(D)$ is the linear map as defined earlier.

The ODE $F(D)y = b(x)$ is called homogeneous ODE if $b(x) = 0$ otherwise it is called nonhomogeneous ODE. By a solution of ODE $F(D)y = b(x)$ we mean a smooth function $y = \varphi(x)$ satisfying $F(D)\varphi(x) = b(x)$. We first study a method to obtain solution of homogeneous ODE.

Solution of Homogeneous ODEs: Let $F(D)y = 0$ be ODE whose solution we seek in the form $y = e^{\lambda x}$ for some scalar λ as a trial. Then observe that $F(D)e^{\lambda x} = F(\lambda)e^{\lambda x} = 0$ which reduces to solving the polynomial equation of degree n in λ namely

$$F(\lambda) = 0$$

and therefore has n roots (Fundamental theorem of Algebra). We have now two cases

Case I: If all roots $\lambda_1, \lambda_2, ..., \lambda_n$ are distinct then the n functions $\varphi_i(x) = e^{\lambda_i x},\ i = 1, 2, ...n$, are solutions of the given homogeneous ODE. Then the $n-$functions defined by $\varphi_i(x) := e^{\lambda_i},\ i = 1, 2, ..., n$ are solutions of $F(D) = 0$; more over the function y_c defined by the linear combination

$$y_c := c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi(x)$$

for arbitrary scalars $c_1, ..., c_n \in \mathbb{R}$ is also a solution of the same ODE and it is called the complementary function of the homogeneous ODE. Here for distinct λ_i's

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$$\boxed{y_c := c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}+\cdots+c_ne^{\lambda_n x}.}$$

Case II: If a root $\lambda=\mu$ of $F(\lambda)=0$ has multiplicity $m>1$ then we write $F(D)=(D-\mu)^m g(D)$ for some polynomial $g(D)$ of degree $n-m\geq 0$ in symbol D. Then $F(D)y=0$ implies $(D-\mu)\{(D-\mu)^{m-1}g(D)y\}=0$ such that

$$(D-\mu)^{m-1}g(D)y_c=c_1e^{\mu x}.$$

Note that

$$(D-\mu)^{m-1}g(D)y_c=(D-\mu)\{(D-\mu)^{m-2}g(D)y_c\}=c_1e^{\mu x}$$

which is a linear ODE for $(D-\mu)^{m-2}g(D)y_c$ and has the solution (proposition 1)

$$(D-\mu)^{m-2}g(D)y_c=e^{\mu x}\left(\int e^{-\mu x}c_1xe^{\mu x}dx+c_2\right)$$

$$\implies\quad (D-\mu)^{m-2}g(D)y_c=(c_1x+c_2)e^{\mu x}.$$

Repeating the same procedure and redefining the arbitrary constants we obtain

$$G(D)y_c=(c_1+c_2x+\cdots+c_mx^{m-1})e^{\mu x}.$$

Note that for any real a, $e^{ax}D^{-1}\{e^{-ax}(c_1+c_2x+\cdots+c_mx^{m-1})e^{\mu x}\}=(c_1'+c_2'x+\cdots+c_m'x^{m-1})e^{\mu x}$ for redefined constants c_i'. This shows that if $(D-\nu)^k, k\geq 1,\ \nu\neq\mu$ is a factor of $g(D)$ such that $g(D)=(D-\nu)^kh(D)$ then

$$h(D)y_c=(c_1+c_2x+\cdots+c_mx^{m-1})e^{\mu x}+(d_1+\cdots+d_kx^{k-1})e^{\nu x}$$

for some arbitrary scalars $d_1,...,d_k$, and so on.
Example: 1 Suppose we want to solve the ODE

$$(D-1)^3(D+2)^2(D+4)y=0$$

then note that here $F(D)=(D-1)^3(D+2)^2(D+4)$ and therefore $F(\lambda)=0$ gives $\lambda=1,1,1,-2,-2,-4$. Consequently in view of **Case I** and **II** above the solution form of this ODE is given by

$$y_c=(c_1+c_2x+c_3x^2)e^x+(c_4+c_5x)e^{-2x}+c_6e^{-4x}.$$

Nonhomogeneous Linear ODEs:
Here we consider the ODE $F(D)y=b(x)$ where $b(x)$ is a smooth real function of $x\in\mathbb{R}$. Let y_c be the solution of $F(D)y_c=0$ and let y_p be a particular solution of

$$F(D)y=b(x)$$

such that

$$\tilde{y}_p:=\widetilde{F(D)}^{-1}(b(x))$$

since $\widetilde{F(D)}$ is invertible and

$$\tilde{y}_p=y_p+\ker F(D).$$

Then y_p is called particular integral of the differential Eq. $F(D)y=b(x)$ and if we define

$$y:=y_c+y_p,\ y_c\in\ker F(D)$$

then $F(D)(y_c+y_p)=F(D)(y_c)+F(D)(y_p)=0+b(x)=b(x)$ which proves that y_c+y_p is a solution of $F(D)y=b(x)$ and it is called the complete solution of the ODE. We now describe the various techniques to obtain $\tilde{y}_p:=\widetilde{F(D)}^{-1}(b(x))$ when $b(x)$ is one among the following types:
Type I: ($b(x)=e^{ax}f(x)$)
Note that $\widetilde{F(D)}(e^{ax}\widetilde{F(D+aI)}^{-1}f(x))=e^{ax}f(x)$ from which it directly follows that

$$\boxed{\widetilde{F(D)}^{-1}(e^{ax}f(x))=e^{ax}\widetilde{F(D+aI)}^{-1}(f(x))}$$

Up to $y_c\in\ker F(D)$ we can identify $\widetilde{F(D)}$ by $F(D)$ so to avoid the cumbersome notation we will write

$$\widetilde{F(D)}\equiv F(D).$$

Type II,III,IV
We refer you to the following table:

1. $\boxed{F(D)^{-1}(xf(x))=xF(D)^{-1}(f(x))-F'(D)(F(D)^{-1})^2(f(x))}$
2. For a positive integer k, $F(D)^{-1}(x^k)=a_j^{-1}(I+g(D))^{-1}(x^k)$ for some nonzero $a_j,\ j=0,...,n$ where
$$g(D)=\frac{a_0}{a_j}D+\cdots+\frac{a_{j-1}}{a_j}D^{j-1}+\frac{a_{j+1}}{a_j}D^{j+1}+\cdots\frac{a_n}{a_j}D^n.$$
Then we have
$$\boxed{(I+g(D))^{-1}(x^k)=(I-g(D)+(g(D))^2-\cdots)(x^k)}$$
which just involves derivatives $D^i,\ i=1,2,...$ of x^k and can be evaluated.
3. $F(D^2)^{-1}(\sin(ax+b))=F(-a^2)^{-1}\sin(ax+b),\ F(-a^2)\neq 0.$ If $F(-a^2)=0$ i.e. if $F(D^2)=(D^2+a^2I)^kg(D^2),\ k\in\mathbb{Z}^+$ such that now $g(-a^2)\neq 0$ then
$$F(D^2)^{-1}(\sin(ax+b))=[(D^2+a^2I)^kg(D)]^{-1}(\sin(ax+b))$$
$$=g(-a^2)^{-1}(D^2+a^2I)^{-k}\sin(ax+b)$$
4. A direct check reveals that
$$(D^2+a^2I)^{-1}\sin(ax+b)=-\frac{x}{2a}\cos(ax+b)$$
5. $(D-aI)^k\left(\frac{x^k}{k!}e^{ax}\right)=e^{ax}D^k\frac{x^k}{k!}=e^{ax}$. It follows that
$$\boxed{(D-aI)^{-k}e^{ax}=\frac{x^k}{k!}e^{ax}}$$
6. Observe that (when we extend canonically the linear map D from real to complex C^∞ functions)
$$(D^2+a^2I)^k\left(\frac{x^k}{k!}(e^{iax}\pm e^{-iax})\right)=(D+iaI)^ke^{iax}+(D-iaI)^ke^{-iax}$$
$$\Rightarrow (D^2+a^2I)^k\left(\frac{x^k}{k!}\sin(ax)\right)=(2i)^{k-1}a^k(e^{iax}-(-1)^ke^{iax})$$
and
$$(D^2+a^2I)^k\left(\frac{x^k}{k!}\cos(ax)\right)=2^{k-1}(ia)^k(e^{iax}+(-1)^ke^{iax})$$
We conclude that
$$(D^2+a^2I)^k\left(\frac{x^k}{(2a)^kk!}\sin(ax)\right)=\begin{cases} i^k\sin(ax) & \text{if } k \text{ is even}\\ i^{k-1}\cos(ax) & \text{if } k \text{ is odd}\end{cases}$$
$$(D^2+a^2I)^k\left(\frac{x^k}{(2a)^kk!}\cos(ax)\right)=\begin{cases} i^k\cos(ax) & \text{if } k \text{ is even}\\ i^{k-1}\sin(ax) & \text{if } k \text{ is odd}\end{cases}$$
Hence we have the following formulas:
$$\boxed{(D^2+a^2I)^{-k}\sin(ax)=\begin{cases} \frac{1}{k!}\left(\frac{x}{2ia}\right)^k\sin(ax) & \text{if } k \text{ is even}\\ \frac{i}{k!}\left(\frac{x}{2ia}\right)^k\cos(ax) & \text{if } k \text{ is odd}\end{cases}}$$
$$\boxed{(D^2+a^2I)^{-k}\cos(ax)=\begin{cases} \frac{1}{k!}\left(\frac{x}{2ia}\right)^k\cos(ax) & \text{if } k \text{ is even}\\ \frac{i}{k!}\left(\frac{x}{2ia}\right)^k\sin(ax) & \text{if } k \text{ is odd}\end{cases}}$$