Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Assignment #1
(Numerical Analysis)

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1. Define error and round off error. If the number x is rounded off to N decimal places then show that $\Delta x = \frac{1}{2}10^{-N}$. Hence calculate $\sqrt{3} + \sqrt{5}$ correct to 4 decimal places.

2. If an observation is represented by a functional dependence $f(x_1, x_2, \cdots, x_n)$ obtain the general error formula in calculation of f, i.e. $\dfrac{\Delta f}{f} = \dfrac{\partial f}{\partial x_1}\Delta x_1 + \dfrac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \dfrac{\partial f}{\partial x_n}\Delta x_n$. Hence obtain maximum relative error in evaluation of $\dfrac{5xy^4}{3+z^2}$ if errors in x, y and z are bounded by .01, .002, and .045, respectively.

3. State Rolle's theorem, Lagrange's mean value theorem and the Taylor's formula for a function of one real variable. Obtain error in the Taylor's formula in expansion of a function $f(x)$ of class C_n about the point $x = a$, i.e.
$$R_n(x) = \frac{(x-a)^n}{n!} f^{(n)}(\xi), \ a < \xi < x.$$
Hence evaluate the value of e^2 correct to 5 decimal places and obtain an error bound for e^2 for this calculation.

4. How many terms are needed in the Maclaurin expansion of $\sin x$ to obtain $\sin 1$ correct to 8 decimal places. [**Hint**: use from Q.1 $\Delta x = \frac{1}{2}10^{-N}$.]

5. Derive the Maclaurin expansion of the function $\tan x$. Using it evaluate π correct to 4 decimal places.

6. State Weirstrass theorem regarding location of a real root of the equation $f(x) = 0$. Discuss the bisection method to obtain a real root of this equation. Hence evaluate the minimum number n of iterations needed to approximate the root with the error at most ϵ i.e. prove that
$$n = \left[\frac{\ln(|b-a|) - \ln(\epsilon)}{\ln(2)}\right]$$
where $[\cdot]$ denotes the greatest integer function, and (a, b) is the interval of the solution.

7. Using bisection method find a real root of the equation $xe^x = 1$ which lies between 0 and 1 within 3% of error.

8. Discuss the convergence of the iterative method in solving $x = \varphi(x)$. Also obtain the Aitken's Δ^2-process to obtain second order convergence of the iterative method i.e. prove that the root at $i-$th iteration is given by $\xi = x_{i+1} - \dfrac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$. Utilize this process in solving $x = (3 + \cos x)/2$ to obatin $x_4 = 1.524$.

9. Discuss the method of false positions to obtain a real root of the equation $f(x) = 0$. Use it to obtain a real root of the equation $x^3 - 2x - 5 = 0$.

10. Prove that if an approximate real root of the equation $f(x) = 0$ lies in the interval $(x_0 - h, x_0 + h)$ for $h > 0$ sufficiently small, the sequence $\{x_n\}$ defined by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ converges to a real root of the equation $f(x) = 0$. Using this obtain a real root of the equation $x^3 - 2x - 5 = 0$.

11. Define rank of a matrix A. Discuss the consistency of the linear system $AX = B$ where $A = (a_{ij})_{m\times n},\ X = (x_j)_{n\times 1},\ B = (b_i)_{m\times 1}$ in terms of the ranks of the matrix A and the augmented matrix $[A|B]$.

12. Using Gauss elimination method, solve the linear system $\begin{array}{c} 2x + y + z = 10 \\ 3x + 2y + 3z = 18 \\ x + 4y + 9z = 16 \end{array}$. Also obtain inverse of the coefficient matrix using elementary row operations.

13. Solve the following system using $LU-$decomposition method (Clairaut's method) $\begin{array}{c} 2x + 3y + z = 9 \\ x + 2y + 3z = 6 \\ 3x + y + 2z = 16 \end{array}$.

14. Discuss the convergence of the iterative-method in solving the linear system $AX = B$. Using Jacobi and Gauss-Seidel methods solve the linear system, to three decimals $\begin{array}{c} 83x + 11y - 4z = 95 \\ 7x + 52y + 13z = 104 \\ 3x + 8y + 29z = 71 \end{array}$.

15. Use iterative method to find the largest eigenvalue and the corresponding eigenvector of the matrix

$$A = \begin{pmatrix} 5 & 2 & 1 & -2 \\ 2 & 6 & 3 & -4 \\ 1 & 3 & 19 & 2 \\ -2 & -4 & 2 & 1 \end{pmatrix}.$$

16. State the general error formula for a polynomial interpolation of a set of $n+1$ tabulated values (x_0, y_0), (x_1, y_1), ...(x_n, y_n) for a function $y = f(x)$ of class C_n.

17. Define the following difference operators $E,\ \Delta,\ \nabla,\ \delta,\ \delta_n$. Prove the Newton forward and backward difference formulas for a polynomial interpolation:

$$(i)\quad f(x+rh) = \left(1 + r\Delta + \frac{r(r-1)}{2!}\Delta^2 + \cdots\right) f(x).$$

$$(ii)\quad f(x-rh) = \left(1 - r\nabla + \frac{r(r+1)}{2!}\nabla^2 + \cdots\right) f(x).$$

18. Gauss forward central difference formula is given by

$$f(x+rh) = f(x) + r\delta_{\frac{1}{2}} + \sum_{n\geq 1} C(r-1+n,\ 2n)\delta_0^{2n} + \sum_{n\geq 1} C(r+n,\ 2n+1)\delta_{\frac{1}{2}}^{2n+1}$$

where $\delta_n = f(x+nh+\frac{h}{2}) - f(x+nh-\frac{h}{2})$ and $C(m,k) = \frac{m(m-1)\cdots(m-k+1)}{k(k-1)\cdots 1}$. Express this formula in terms of the forward differences Δ^n.

19. Find a cubic polynomial satisfying the values: $y(1) = 24,\ y(3) = 120,\ y(5) = 336,\ y(7) = 720$. Hence obtain $y(8)$.

20. Using Newton's forward difference formula find the sum $S_n = 1^3 + 2^3 + \cdots n^3$. [**Hint**: $\Delta S_n = (n+1)^3$, $\Delta^2 S_n = \Delta S_{n+1} - \Delta S_n$, etc. Then $S_n = S_{1+n-1} = \left(1 + (n-1)\Delta + \frac{(n-1)(n-2)}{2}\Delta^2 + \cdots\right) S_1$.]

21. Find the missing term in the following table:

x :	0	1	2	3
y :	3	9	-	81

Explain why result differs from $3^3 = 27$?

22. Locate and correct the error in the following table:

x :	2.5	3.0	3.5	4.0	4.5	5.0	5.5
y :	4.32	4.83	5.27	5.47	6.26	6.79	7.23

23. State Stirling's formula and Bessel's formula for interpolation. Using these formulas evaluate $e^{1.91}$ from the following table:

x :	1.7	1.8	1.9	2.0	2.1	2.1
e^x :	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

24. State Gauss backward central difference formula. Using it find $\sqrt{12526}$ given that $\sqrt{12500} = 111.803399$, $\sqrt{12510} = 111.848111$, $\sqrt{12520} = 111.892806$, $\sqrt{12530} = 111.937483$.

25. Derive the Lagrange's interpolation formula. Given the tabular values:

x :	-2	-1	2	3
$f(x)$:	-12	-8	3	5

obtain a cubic polynomial using Lagrange's interpolation.

26. Using Lagrange's formula express the rational function $\dfrac{3x^2+x+1}{(x-1)(x-2)(x-3)(x-4)}$ as a sum of partial fractions.

27. Derive the Newton's divided difference formula

$$y(x) = y(x_0) + (x-x_0)[x_0, x_1] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})[x_0, x_1, \cdots, x_n] + R_{n+1}(x)$$

where $R_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)[x, x_0, x_1, \cdots, x_n]$.

28. Given the set of tabulated points $(1,-3),\ (3,9),\ (4,30)$, and $(6,132)$ obtain the value of $y(x=2)$ using Newton's divided difference formula.

29. If $f(x) = \dfrac{1}{x},\ x \neq 0$ prove that $[x_0, x_1, \cdots, x_r] = \dfrac{(-1)^r}{x_0 x_1 \cdots x_r}$.