Notes on Limit, Continuity, and Differentiability

Jitender Singh
Department of Mathematics, Guru Nanak Dev University, Amritsar-143005, India
E-mail: sonumaths@gmail.com

Definition 1(Limit): A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a limit ℓ at a point $x = a,\ a \in \mathbb{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - \ell| < \epsilon$$

Note that in this definition, $\delta = \delta(\epsilon)$ i.e. δ is a function of ϵ.
We may define left hand limit (LHL) and right hand limit (RHL) of f at point $x = a$ to be as follows:
Definition 2(LHL): $f(x)$ is said to have left hand limit ℓ_1 at point $x = a$ if for every $\epsilon > 0$, there is a $\delta_1 > 0$ such that

$$a - x < \delta_1 \quad \Rightarrow \quad |f(x) - \ell_1| < \epsilon$$

Definition 3(RHL): $f(x)$ is said to have right hand limit ℓ_2 at point $x = a$ if for every $\epsilon > 0$, there is a $\delta_2 > 0$ such that

$$x - a < \delta_2 \quad \Rightarrow \quad |f(x) - \ell_2| < \epsilon$$

Definition 4: $f(x)$ is said to be continuous at a point $x = a$ of its domain if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon$$

Definition 5: $f(x)$ is said to be differentiable at a point $x = a$ of its domain with derivative $f'(a) \in \mathbb{R}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon$$

Remark 1: If left hand Limit of a function f at a point $x = a$ is ℓ_1 we write $\lim\limits_{x\to a^-} f(x) = \ell_1$. Similarly if right hand Limit of f at $x = a$ is ℓ_2 then we write $\lim\limits_{x\to a^+} f(x) = \ell_2$. If Limit of f at a point $x = a$ is ℓ we write $\lim\limits_{x\to a} f(x) = \ell$. If f is continuous at $x = a$, we write $\lim\limits_{x\to a} f(x) = f(a)$. and if f is differentiable at $x = a$ with derivative $f'(a)$, we write $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$.
Theorem 1: If a function f is differentiable at a point $x = a$ with derivative $f'(a)$ then f is continuous at $x = a$.
Proof: Let f is differentiable at a point $x = a$ with derivative $f'(a)$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - a| < \delta_1 \quad \Rightarrow \quad \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon$$

As

$$\left|\frac{f(x) - f(a)}{x - a}\right| - |f'(a)| \le \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon,$$

we need to find a $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| \le (\epsilon + |f'(a)|)|x - a| < \epsilon.$$

This can be done by defining $\delta := \min\left\{\delta_1, \dfrac{\epsilon}{\epsilon + |f'(a)|}\right\}$. Result follows now.
Remark 2: Converse of the theorem 1 is not true as there are continuous functions which are not differentiable. An example is $f(x) = |x|$ which is continuous at $x = 0$ but fails to have derivative at $x = 0$ as can be seen readily from the following: $\lim\limits_{x\to 0} \dfrac{|x| - 0}{x - 0} = \lim\limits_{x\to 0} \dfrac{|x|}{x}$ which does not exist!
Theorem 2(Chain Rule): If $f : A \to f(A)$ and $g : f(A) \to C$ are differentiable at $x = a$ and $f(x) = f(a)$ respectively, then $g \circ f : A \to C$ is differentiable at $x = a$ and $(g \circ f)'(a) = g'(f(a))f'(a)$.
Proof: Given that for every $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exist $\delta_1,\ \delta_2 > 0$ such that

$$|x - a| < \delta_1 \quad \Rightarrow \quad \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon_1 \tag{0.1}$$

and

$$|f(x) - f(a)| < \delta_2 \quad \Rightarrow \quad \left|\frac{g(f(x)) - g(f(a))}{f(x) - f(a)} - g'(f(a))\right| < \epsilon_2. \tag{0.2}$$

Let $\epsilon > 0$ be given. Consider

$$\begin{aligned}
\left|\frac{(g\circ f)(x) - (g\circ f)(a)}{x-a} - g'(f(a))f'(a)\right| &= \left|\left(\frac{g(f(x)) - g(f(a))}{f(x) - f(a)}\right) \times \left(\frac{f(x) - f(a)}{x-a}\right) - g'(f(a))f'(a)\right| \\
&= \left|\left(\frac{g(f(x)) - g(f(a))}{f(x) - f(a)} - g'(f(a))\right)\frac{f(x) - f(a)}{x-a} + g'(f(a))\frac{f(x) - f(a)}{x-a} - g'(f(a))f'(a)\right| \\
&\le \left|\frac{g(f(x)) - g(f(a))}{f(x) - f(a)} - g'(f(a))\right|\left|\frac{f(x) - f(a)}{x-a} - f'(a) + f'(a)\right| + |g'(f(a))|\left|\frac{f(x) - f(a)}{x-a} - f'(a)\right| \\
&\le \epsilon_2(\epsilon_1 + |f'(a)|) + |g'(f(a))|\epsilon_1 \quad (\textbf{by } (0.1) \textbf{ and } (0.2)) \\
&< \max(\epsilon_1, \epsilon_2)\left\{\max(\epsilon_1, \epsilon_2) + |f'(a)| + |g'(f(a))|\right\} := \epsilon
\end{aligned} \tag{0.3}$$

for

$$\max(\epsilon_1, \epsilon_2) := \frac{1}{2}\left(-|f'(a)| - |g'(f(a))| + \sqrt{(|f'(a)| + |g'(f(a))|)^2 + 4\epsilon}\right) > 0$$

and

$$|x - a| < \delta := \min\{\delta_1, \delta_2\}.$$