Second Derivative Test for Local Extrema

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Let $z = f(x, y)$ be a sufficiently smooth map $f : D_f \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ from an open subset D_f of the real Euclidian plane to the real line. Suppose further that f has a Taylor series expansion of order at least two in D_f. Then for any point $(a, b) \in D_f$ there is an open disc $B((a, b), r) \subset D_f$ of radius r and center at (a, b). Within this disc f has a Taylor series expansion about the point (a, b) as follows:

$$\Delta f = f(a+h, b+k)-f(a, b) = (hf_x+kf_y)(a, b)+\frac{1}{2}(h^2 f_{xx}+2hkf_{xy}+k^2 f_{yy})(a, b)+\mathcal{O}(h^3+k^3). \quad (0.1)$$

If the point (a, b) is an extreme point then to the lowest order the sign of Δf has to be fixed and since it depends only upon the sign of $(hf_x + kf_y)(a, b)$ which may not always be fixed for all (h, k) in a neighborhood of the point $(0, 0)$, the only possibility for the point (a, b) to be a candidate for local extrema is $f_x = f_y = 0$ at (a, b). As a result of this we have

$$\Delta f = \frac{k^2}{2}(At^2 + 2Bt + C) + \mathcal{O}(h^3 + k^3) = \frac{k^2}{2} g(t) + \mathcal{O}(h^3 + k^3) \quad (0.2)$$

where

$$t = \frac{h}{k};\; A = f_{xx}(a, b);\; B = f_{xy}(a, b);\; C = f_{yy}(a, b);\; g(t) = At^2 + 2Bt + C.$$

To the second order, we see that $\text{sign}(\Delta f) = \text{sign}(g(t))$. Therefore f has local maximum (respectively local minimum) at point (a, b) if $\Delta f < 0$ (respectively) $\Delta f > 0$ i.e. $g < 0$ (respectively) $g > 0$ for all points in the neighborhood of (a, b). If $f > 0$ as well as $f < 0$ in every neighborhood of (a, b) then the point (a, b) is called saddle point. We can not conclude anything if $\Delta f = 0 = g(t)$ in a neighborhood of (a, b).

Note that in any neighborhood of $(0, 0)$ quadratic polynomial $Ag(t) = (At+B)^2+AC-B^2 \gtreqless 0$ iff $B^2 - AC \lesseqgtr 0$. Also g has no sign for its unique extremum value $(AC - B^2)/A = 0$ where $g'(t) = 0$ and it corresponds to $t = -B/A$. Consequently, the test in the above context is inconclusive for $AC - B^2 = 0$.

References:

[1] Tom M. Apostol. Calculus II

[2] MIT Notes.

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