THE INCLUSION EXCLUSION PRINCIPLE

JITENDER SINGH
Indian Statistical Institute, Kolkata-700108

In this discussion, we shall be using the following version of the well known inclusion-exclusion principle, to obtain certain probabilities.

PROPOSITION 1 (INCLUSION-EXCLUSION PRINCIPLE). *For a countable collection $(A_\alpha)_{\alpha\in I}$ of sets,*

$$P(A_1 \cup A_2 \cup \cdots) = \sum_{i\geqslant 1} P(A_i) - \sum_{i<j\geqslant 2} P(A_i \cap A_j) + \cdots$$

In this note, we shall address the following question: Given n letters and they are to put in n addressed envelopes at random. The probability that at least one letter should be in the correct envelope is equal to the nice expression

$$1 - \frac{1}{2!} + \cdots + (-1)^{n-1}\frac{1}{n!}, \tag{0.1}$$

which converges to $1 - e^{-1}$ as n approaches infinity. The problem of putting a finite number of letters in a given number of envelopes will be called as "the Letter-Envelope problem".

To our convenience, we define for $k = 1, 2, 3, ...$, the event A_k consisting of all outcomes such that the $k-$th letter occupies correct envelope. Let B_m for $m = 1, 2, \cdots, n$, denote the set of all outcomes such that at least m letters are in correct envelopes. It is easy to see that

$$B_m = A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_m},$$

where $j_1,\ j_2, \cdots, j_m$ are any distinct m members of the set $\{1, 2, \cdots, n\}$. Therefore using the inclusion-exclusion principle, the corresponding probability that at least m letters are in the correct envelopes is given by

$$\begin{aligned} P(B_m) &= \frac{{}^nC_m(n-m)!}{n!} - \frac{{}^nC_{m+1}(n-m-1)!}{n!} + \cdots + (-1)^{n-m}\frac{{}^nC_n 0!}{n!}. \\ &= \frac{1}{m!} - \frac{1}{(m+1)!} + \cdots + (-1)^{n-m}\frac{1}{n!}, \end{aligned} \tag{0.2}$$

If n letters are to put in k $(n \leqslant k)$ envelopes at random such that n envelopes are addressed and rest of the $k-n$ envelopes are left unaddressed, one obtains in general

$$P(B_m) = \frac{{}^nC_m(k-m)!}{k!} - \frac{{}^nC_{m+1}(k-m-1)!}{k!} + \cdots + (-1)^{n-m}\frac{{}^nC_n(k-n)!}{k!}. \tag{0.3}$$

Note that $P(B_n) = \dfrac{(k-n)!}{k!}$. Also $P(B_m \sim B_{m+1}) = P(B_m) - P(B_{m+1})$ gives the probability that exactly m letters are in the correct envelopes. Similarly we note that $1 - P(B_{m+1})$ is the probability that at most m letters are in correct envelopes; in particular $1 - P(B_1)$ is the probability that none of the letters is in the correct envelope.