Department of Mathematics

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Assignment #2

(Surfaces in $\mathbb{R}^3$)

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1. Define a simple sheet of a surface. Show that the unit sphere is not a simple sheet of a surface.
2. Define a topological space and the intrinsic topology. Define an ordinary surface in $\mathbb{R}^3$ as a topological space.
3. What do you mean by a regular surface of class C_n? Prove that under transformation of curvilinear coordinates $(u^1, u^2) \to (u^{1'}, u^{2'})$, the parametric surface $\mathbf{r} = \mathbf{r}(u^1, u^2)$ can have new singular points in the new coordinate system $(u^{1'}, u^{2'})$ i.e. $\frac{\partial \mathbf{r}}{\partial u^{1'}} \times \frac{\partial \mathbf{r}}{\partial u^{2'}} = \frac{\partial(u^1, u^2)}{\partial(u^{1'}, u^{2'})} \frac{\partial \mathbf{r}}{\partial u^1} \times \frac{\partial \mathbf{r}}{\partial u^2}$. Obtain the singular points of the unit-sphere with center at the origin, under transformation from cartesian to spherical polar coordinates.
4. What do you mean by a differential manifold ? Prove that the unit sphere is a differential manifold.
5. Define an immersion and embedding in manifolds. Give one example of each.
6. Consider the implicit equation $F(x, y, z) = 0$ where F is a smooth function and $\frac{\partial F}{\partial x} \neq 0$ in some neighborhood U of a point P in $\mathbb{R}^3$ such that $F(P) = 0$. Prove that the set $S = \{Q \mid Q \in \overline{U},\ F(Q) = 0\}$ is a simple sheet of a surface.
7. Define a tangent plane and normal vector to a parametric surface at a regular point. Obtain their equations.
8. Find the intersection points of the helicoid $x = u\cos v,\ y = u \sin v,\ z = v$ with the straight line $x = 1,\ y = 0$ and the angle between this line and the surface of the points of intersection. Angle between a curve and a surface is defined as the angle between the tangent to the curve and the tangent plane to the surface.
9. Prove that the angle ϕ between the surface $\mathbf{r} = \mathbf{r}(u^1, u^2)$ and the curve $\mathbf{r} = \mathbf{p}(t)$ at a common point P which corresponds to the value t_0, u_0^1, u_0^2 of the parameters is given by $\sin\phi = \dfrac{\left|\frac{\partial \mathbf{r}}{\partial u^1} \times \frac{\partial \mathbf{r}}{\partial u^2} \cdot \dot{\mathbf{p}}\right|}{\left|\frac{\partial \mathbf{r}}{\partial u^1} \times \frac{\partial \mathbf{r}}{\partial u^2}\right| |\dot{\mathbf{p}}|}$.
10. Prove that a surface all of whose normals intersect a given straight line is a surface of revolution. More over all surfaces of revolutions have this property.
11. Prove that a surface all of whose tangent planes are parallel to a straight line is a cylindrical surface.
12. Prove that if all tangent planes of a surface go through a common point then the surface is a conical surface.
13. Find envelopes of the following family of spheres: $x^2 + y^2 + (z - \alpha)^2 = 2\alpha$.
14. Define a developable surface. Prove that conical and cylindrical surfaces are developable.
15. Consider a curve on the surface $\mathbf{r} = \mathbf{r}(u^1, u^2)$ given by the parametric equations $u^1 = u^1(t),\ u^2 = u^2(t),\ a \le t \le b$. Then prove that arc length between $t = a$ and $t = b$ of the curve is equal to $s = \int_a^b \sqrt{g_{ij} du^i du^j}$ where $g_{ij} = \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial \mathbf{r}}{\partial u^j}$
16. Prove that the first fundamental form of the surface $x = a \sin u^1 \cos u^2,\ y = a \sin u^1 \sin u^2,\ z = a \left(\cos u^1 + \ln \frac{u^1}{2} \right)$ is given by $ds^2 = a^2 \cot^2 u^1 (du^1)^2 + a^2 \sin^2 u^1 (du^2)^2$.
17. Obtain the first and second fundamental forms of the surface of revolution $x = f(u)\cos v,\ y = f(u)\sin v,\ z = h(u)$.
18. Prove that the angle between two curves on a surface at a common point is given by $\cos\phi = \dfrac{g_{ij} du^i \delta u^j}{\sqrt{g_{ij} du^i du^j} \sqrt{g_{ij} \delta u^i \delta u^j}}$ where du^i and δu^i are the differentials in the directions of the two curves respectively at the common point. In particular if the curves are just u^1-lines and u^2-lines then we can choose $du^1 = 1$ $du^2 = 0$, and $\delta u^1 = 0$, $\delta u^2 = 1$. Consequently the angle between these lines is $\cos\omega = \frac{g_{12}}{\sqrt{g_{11} g_{22}}}$. Hence the coordinate lines are orthogonal if and only if $g_{12} = 0$.
19. Prove that the surface-area of the parametric surface $\mathbf{r} = \mathbf{r}(u^1, u^2)$ over a rectangle Ω is given by $\int_\Omega \sqrt{g} du^1 du^2$. Hence obtain the surface-area of the torus.
20. Using the second fundamental form of a surface $b_{ij} du^i du^j$, classify the points on a surface as elliptic, parabolic and flat points. Obtain parabolic point of the surface $z = x^3 - y^2$ discuss the deviation of the surface from the tangent plane at this point.
21. Show that all points of a ruled surface which is not developable are hyperbolic.
22. Define the Gaussian curvature K of a surface and prove that $K = \frac{b}{g} = \frac{b_{11} b_{22} - (b_{12})^2}{g_{11} g_{22} - (g_{12})^2}$ i.e. the ratio of the discriminants of the second and first fundamental form respectively.
23. Consider a sufficiently smooth parametric surface $\mathbf{r} = \mathbf{r}(u^1, u^2)$. Prove that $\frac{\partial^2 \mathbf{r}}{\partial u^i u^j} = \Gamma^k_{ij} \frac{\partial \mathbf{r}}{\partial u^k} + b_{ij}\mathbf{m}$ where $\mathbf{m} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|}$. Also prove that the components $\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{jm}}{\partial u^i} + \frac{\partial g_{mi}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^m} \right)$.