Department of Mathematics

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Assignment #1

(General–Curve Theory)

January 22, 2010

1. Define the following terms and give two examples for each: (i) locally injective map (ii) curve and a parametric curve (iii) equivalent-paths (iv) homeomorphism of maps (v) a function of class C_n (vi) regular point and singular points of a curve (vii) rectification point of a curve (viii) regular parametric curve
2. Define a locally injective map. Give an example of a map that is locally injective but not injective.
3. Prove the following:
   (a) if a vector-valued function $\mathbf{r}(t)$ has a continuous derivatives up to the order $n+1$ inclusively, then
   $$\mathbf{r}(t_0+h) = \mathbf{r}(t_0) + \frac{h}{1!}\dot{\mathbf{r}}(t_0) + \frac{h^2}{2!}\ddot{\mathbf{r}}(t_0)\cdots + \frac{h^n}{n!}\mathbf{r}^{<n>}(t_0) + h^{n+1}\mathbf{U}_{n+1}(h)$$
   where the function $\mathbf{U}_{n+1}(h)$ is bounded in some neighborhood of $h=0$.
   (b) the equation $\mathbf{r}(t) = t\mathbf{a}_0 + \mathbf{b}$ for $t \in \mathbb{R}$ and constant vectors $\mathbf{a}_0$ and $\mathbf{b}$, represent a curve.
   (c) a circle is a curve.
   (d) the semicubical parabola $x = t^2,\ y = t^3,\ z = 0,\ -\infty < t < \infty$ has an essential singularity $(0,0,0)$.
   (e) lengths of equivalent paths are equal.
   (f) a regular piecewise parametric representation $\mathbf{r} = \mathbf{r}(t),\ a \le t \le b$ is rectifiable and its length between the points on the curve corresponding to $t = a$ and $t = t$, is given by
   $$s(t) = \int_a^t |\dot{\mathbf{r}}(\theta)|\, d\theta.$$
   Further, $s(t)$ is a monotonically increasing, continuous and differentiable function of t.
   (g) The natural representation of a parametric curve has a singular point if and only if this point is an essential singularity of the curve.
4. Trace the curves $\mathbf{r} : (0,\pi] \to \mathbb{R}^2$ defined by $\mathbf{r}(t) = (\cos nt \cos t,\ \cos nt \sin t,\ 0)$ where n is a positive integer.
5. Define contact of two parametric curves. Prove that two parametric curves $\mathscr{K}_1$, and $\mathscr{K}_2$ of class C_{n+1} have a contact of order n at a common point P if and only if their natural representations $\mathbf{r} = \mathbf{r}_1(s)$ and $\mathbf{r} = \mathbf{r}_2(\sigma)$ satisfy the following at P $\mathbf{r}_1 = \mathbf{r}_2,\ \mathbf{r}_1^{<i>} = \mathbf{r}_2^{<i>}$ for all $1 \le i \le n$ and $\mathbf{r}_1^{<n+1>} \ne \mathbf{r}_2^{<n+1>}$.
6. Determine order of contact of the straight line $\mathbf{R}(u) = (1-u)\mathbf{r}(t) + u\dot{\mathbf{r}}(t)$ and the circle $\mathbf{r} = (\cos t,\ \sin t,\ 0),\ 0 \le t < 2\pi$; at the two common points P and Q corresponding to the parameter values $u = 0,\ 1$, respectively.
7. Determine order of contact of the straight line $\mathbf{R}(u) = (1+u)\mathbf{r}(t) - u(0,0,1)$ and the circle $\mathbf{r} = (\cos t,\ \sin t,\ 0),\ 0 \le t < 2\pi$; Where $u \in \mathbb{R}$.
8. Prove that the tangents to the helix at various points form a constant angle with the $z-$axis.
9. Find a curve on the cone $x^2 + y^2 - z^2 = 0$ which intersects the straight lines on the cone with a constant angle α. Prove that the tangents to the curve form a constant angle with the $z-$axis.
10. Find the osculating plane of the curve $x = t^2,\ y = t - t^2,\ z = 2t$.
11. Find osculating plane of the helix. What is the angle between it and the axis of the helix ?
12. Derive the Serret-Frenet's formula for a parametric curve in $\mathbb{R}^3$ i.e. prove that for a natural representation $\mathbf{r} = \mathbf{r}(s)$ of a parametric curve of class at least C_3, at a regular point corresponding to the parameter value s which is not a rectification point, the following holds:
   $$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$
   where $\kappa = |\mathbf{r}''|$ and $\tau = \mathbf{b}\cdot\mathbf{n}'$.
13. If κ and τ are independent of s in the Serret-Frenet formula then prove that
   $$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}(s) = \begin{pmatrix} \dfrac{\tau^2 + \kappa^2\cos\{\sqrt{\kappa^2+\tau^2}s\}}{\kappa^2+\tau^2} & \dfrac{\kappa\sin\{\sqrt{\kappa^2+\tau^2}s\}}{\sqrt{\kappa^2+\tau^2}} & -\kappa\tau\dfrac{\cos\{\sqrt{\kappa^2+\tau^2}s\}-1}{\kappa^2+\tau^2} \\ -\dfrac{\kappa\sin\{\sqrt{\kappa^2+\tau^2}s\}}{\sqrt{\kappa^2+\tau^2}} & \cos\{\sqrt{\kappa^2+\tau^2}s\} & \dfrac{\tau\sin\{\sqrt{\kappa^2+\tau^2}s\}}{\sqrt{\kappa^2+\tau^2}} \\ -\kappa\tau\dfrac{\cos\{\sqrt{\kappa^2+\tau^2}s\}-1}{\kappa^2+\tau^2} & -\dfrac{\tau\sin\{\sqrt{\kappa^2+\tau^2}s\}}{\sqrt{\kappa^2+\tau^2}} & \dfrac{\kappa^2+\tau^2\cos\{\sqrt{\kappa^2+\tau^2}s\}}{\kappa^2+\tau^2} \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}(0)$$
14. Find the Frenet frame of the helix. What is the angle between $\mathbf{t},\ \mathbf{n},\ \mathbf{b}$ and the positive axis of the helix ?
15. Find a function $\phi(u)$ such that the principal normals of the curve $x = a\cos u,\ y = a\sin u,\ z = \phi(u)$ are parallel to the $xy-$plane. What kind of curve is this ?
16. Prove that all osculating planes of the helix which go through a common point which does not lie on the helix itself are tangent to the helix at points which line in one plane.
17. Prove that if all the normal planes of a curve of class C_1 have a point in common then the curve lies on a sphere with center at this point.