Derivative of a Determinant

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Let $a_{ij},\ A^{ij}$ be components of two $n \times n$ $(n \in \mathbb{Z}^+)$ matrices and the determinant $a := \sum_k a_{ik}A^{ki}$ such that A^{ki} is the cofactor of a_{ik}. Suppose further that the matrix components a_{ij} are smooth functions of curvilinear coordinates $\{x^i\}$. Then for a fixed i

$$\frac{\partial a}{\partial a_{ij}} = \sum_k \frac{\partial a_{ik}}{\partial a_{ij}} A^{ki} + \sum_k a_{ik} \frac{\partial A^{ki}}{\partial a_{ij}} = \sum_k \delta^k_j A^{ki} = A^{ji}$$

since $\dfrac{\partial A^{ki}}{\partial a_{ij}} = 0$ as the cofactor A^{ki} of a_{ik} is not a function of a_{ij} for a fixed i and all j. Therefore in a system of curvilinear coordinates $\{x^i\}$,

$$\frac{\partial a}{\partial x^q} = \frac{\partial a}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial x^q} = A^{ji} \frac{\partial a_{ij}}{\partial x^q}.$$

Under a change of curvilinear coordinates $\{x^i\} \to \{\tilde{x}^i\}$ one has

$$\frac{\partial \tilde{a}}{\partial \tilde{x}^q} = \tilde{A}^{ji} \frac{\partial \tilde{a}_{ij}}{\partial \tilde{x}^q} = T^j_\ell T^i_m S^p_q A^{\ell m} \frac{\partial}{\partial x^p} \left(S^\alpha_i S^\beta_j a_{\alpha\beta} \right) = T^j_\ell T^i_m S^p_q A^{\ell m} \left(a_{\alpha\beta} \frac{\partial}{\partial x^p} (S^\alpha_i S^\beta_j) + S^\alpha_i S^\beta_j \frac{\partial a_{\alpha\beta}}{\partial x^p} \right).$$

This simplifies to

$$\frac{\partial \tilde{a}}{\partial \tilde{x}^p} = S^p_q \left(\frac{\partial a}{\partial x^p} + T^j_\ell T^i_m A^{\ell m} a_{\alpha\beta} \frac{\partial}{\partial x^p} (S^\alpha_i S^\beta_j) \right)$$

where $S = T^{-1}$ and S and T are the linear transformations for the change of basis formula in a linear space V of finite dimension n i.e. with respect to a change of basis from $\{e_i\} \to \{\tilde{e}_i\}$ in V we have the following:

$$\tilde{e}_i = S^j_i e_j,\ e_i = T^j_i \tilde{e}_j$$

which with a slight modification generalizes to define a tensor X of rank $r+s$ ($r-$times contravariant and $s-$times covariant) whose components $X^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ satisfy under a change of basis $\{e_i\} \to \{\tilde{e}_i\}$ the following transformation rule

$$X^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = S^{i_1}_{h_1} \cdots S^{i_r}_{h_r} T^{k_1}_{j_1} \cdots T^{k_s}_{j_s} \tilde{X}^{h_1 h_2 \cdots h_r}_{k_1 k_2 \cdots k_r}$$

and the reverse transformation rule

$$\tilde{X}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = T^{i_1}_{h_1} \cdots T^{i_r}_{h_r} S^{k_1}_{j_1} \cdots S^{k_s}_{j_s} X^{h_1 h_2 \cdots h_r}_{k_1 k_2 \cdots k_r}.$$

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