A version of the closed graph theorem

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Abstract

In this note we discuss a variation of the closed graph theorem given by my students Ms. Kirandeep Kaur and Ms. Rupinder Kaur.

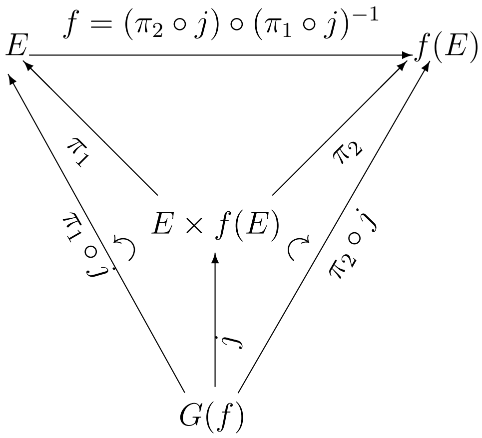
We need the following simple lemma:

Lemma: A continuous bijective map from a compact topological space to a Hausdorff space is a homeomorphism.

Proof: Let $f : X \to Y$ be a continuous bijective map such that X is compact and Y be Hausdorff. Then for any closed subset A of X, A is compact as being closed subspace of a compact space. Consequently $f(A)$ is compact in Y by continuity of f. But then $f(A)$ is closed in Y as compact subspace of a Hausdorff space is closed. This proves f^{-1} is continuous.

Theorem: Let E be a compact Hausdorff space and $f : E \to f(E)$ be a map such that the graph $G(f) := \{x \times f(x) \mid x \in E\}$ of f is a subspace of the product space $E \times f(E)$. Then f is continuous if and only if $G(f)$ is compact.

Proof: Observe the following commuting diagram:



Here π_1 and π_2 are the projection maps from the product space $E \times f(E)$ onto its components which are always continuous. Also, j denotes the inclusion map from the subspace $G(f)$ into $E \times f(E)$ which is also continuous in the subspace topology. From the diagram it is clear that $f = (\pi_2 \circ j) \circ (\pi_1 \circ j)^{-1}$ where the map $\pi_1 \circ j$ is a bijection and f is continuous if and only if the inverse map $(\pi_1 \circ j)^{-1}$ is continuous if and only if $G(f)$ is compact (because if $G(f)$ is compact then $(\pi_1 \circ j)^{-1}$ is continuous by the preceding lemma. Conversely if $(\pi_1 \circ j)^{-1}$ is continuous then it is a homeomorphism and $G(f)$ being homeomorphic image of the compact space E is compact.)

References:

[1] J. Munkres. Topology, John-Wiley, 2005.

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