Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Assignment #3 (Group-Theory)

1. Prove the following:
   (a) If $(ab)^n = a^n b^n$ for all $n > 2$ and $n \in \mathbb{Z}$, $a, b \in G$, then G is abelian.
   (b) Let m and n be two positive integers such that $mr + ns = 1$ for some integers r and s. Suppose $x^t y^t = y^t x^t$, $t = m, n$ for all $x, y \in G$. Then $(xy)^{mr} = (yx)^{mr}$ and $(xy)^{ns} = (yx)^{ns}$ for all $x, y \in G$. Deduce that G is abelian.
   (c) Let $|G| < \infty$, then for all $x \in G$ $|x|$ divides $|G|$.
   (d) There exists an injective homomorphism from the Klein-4 group V_4 to the dihedral group D_8.
   (e) There exists an injective homomorphism from the group D_8 to the group S_4.
   (f) The additive groups $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ are not cyclic.
   (g) The groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \times)$ are isomorphic.
   (h) Every $p-$group has a nontrivial center.
   (i) Let $N \leq G$ then $N \trianglelefteq G \iff G/N$ is a group under the binary operation $xN * yN = xyN$.
   (j) Groups of orders 12, 30, and 385 are not simple.
   (k) Let $|G| = pqr, p < q < r$, where p, q, r, are prime numbers. Then G is not simple.
   (l) $A_4 \ncong D_{12}$ i.e. there exist at least two non-isomorphic non-abelian groups of order 12.
2. Give an example to show that if $H \trianglelefteq K \trianglelefteq G$ then it is not always true that $H \trianglelefteq G$.
3. Let H and K be subgroups of a finite group G. Prove that $|HK| = \dfrac{|H||K|}{|H \cup K|}$.
4. Let $\mathbb{Z}_4 \times \mathbb{Z}_4 := \left\langle x, y \mid x^4 = y^4 = 1,\ xy = yx \right\rangle$; Show that $|H| = 8$ and that $G/\left\langle x^2y^2 \right\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.
5. *Second Isomorphism theorem:* Let A, B be subgroups of G s.t. $A \leq N_G(B)$. Then prove that
   (a) $AB \leq G$; $B \trianglelefteq AB$; $(A \cap B) \trianglelefteq A$
   (b) $AB/B \cong A/(A \cap B)$
6. State and prove the third isomorphism theorem.
7. (a) *Jordan-Hölder theorem:* Prove that every finite group possesses a composition series. The two composition series are equivalent in the sense that both have same number of composition factors and the composition factors in them are isomorphic up to a permutation on number of symbols equal to the length of each composition series.
   (b) Obtain two composition series for $\mathbb{Z}/30\mathbb{Z}$ and compare the composition factors.
   (c) Write a composition series for $S_n, n \geq 3$. Deduce that S_n can not have a composition series of length 3.
   (d) Write a composition series for $D_{2n}, n \geq 3$. When can D_{2n} have a composition series of length 3.
   (e) Prove that a group of order pq, p, q being distinct primes, is solvable.
8. Let $|G| = pq$, $p < q$ and p, q are primes. Prove that if $p \nmid q - 1$, then G is cyclic. Prove also that if $p | q - 1$ then there exists a nonabelian group of order pq.
9. Let $|G| = p_1 p_2 \cdots p_k$ be product of k distinct primes $p_i, 1 \leq i \leq k$ s.t. $p_1 < p_2 < \cdots < p_k$. If for every $1 \leq i \leq k - 1$, $p_i \nmid p_j - 1$ for all $i < j \leq k$, then prove that G is cyclic.
10. *Cauchy's theorem* Let G be a finite group. If a prime number p divides $|G|$, prove that there is a subgroup of G of order p.
11. Prove the following *Sylow theorems*:
    (a) Let G be a finite group. Prove that if there is a prime p and a positive integer α such that $p^\alpha \big| |G|$ then G has a subgroup of order p^α. In particular G has an element of order p.
    (b) Prove that any two Sylow-p-subgroups of a finite group G are conjugates.
    (c) Prove that the number of Sylow-p-subgroups of a finite group G is n_p such that $n_p \equiv 1 \bmod p$. Further if H_p denotes a Sylow-p-subgroup of G then $n_p = |G : N_G(H_p)|$.
    (d) Deduce the following formula:
    $|G : H_p| = |G : N_G(H_p)||N_G(H_p) : H_p|$.
12. (a) Every group is isomorphic to some subgroup of a permutation group.
    (b) Let G be a finite group and $H \leq G$ be such that $|G : H| = p$ where p is the smallest prime divisor of $|G|$. Then prove that $H \trianglelefteq G$.
    (c) Deduce that if $H \leq G$ such that $|G : H| = 2$ then $H \trianglelefteq G$.
13. Prove the following:
    (a) Any $\sigma \in S_n$ can be written as a product of disjoint cycles.
    (b) The alternating group A_5 is simple.
    (c) The alternating group A_n is simple for all $n \geq 6$.
14. Derive the expressions for $|Z(S_n)|$ and $|Z(A_n)|$.
15. Prove that $\mathrm{Aut}(S_3) \cong S_3$; $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.
16. Prove that $H \mathrm{char} K$ and $K \trianglelefteq G$ implies $H \trianglelefteq G$.
17. (a) Prove that if $N \trianglelefteq G$ and N and G/N are solvable, then G is solvable.
    (b) A subgroup of a solvable group is solvable.
    (c) Prove that a finite group G is solvable if and only if there exists a normal series for G with cyclic factors of prime orders.
    (d) Every abelian group is solvable.
    (e) Define a commutator subgroup G' of a group G. Show that $G^{(n)} \trianglelefteq G$ for all $n \in \mathbb{Z}^+$. Prove that G is solvable if and only if $G^{(n)} = 1_G$ for some positive integer n. In particular deduce that every abelian group is solvable.
    (f) Write a solvable series for a group of order 243.
18. Construct a group of order 72 whose center is of order 18.
19. Construct a group of order 36 whose center is of order 6.
20. Let $H \leq \mathbb{R}$ under the operation of usual addition $(+)$. Prove that either H is *closed* in $\mathbb{R}$ or H is *dense* in $\mathbb{R}$.

Reference Book: David. S. Dummit and Richard M. Foote. *Abstract Algebra*, Wiley Indian Reprint (2008).

[1]E-mail: sonumaths@gmail.com; Web page: https://sites.google.com/site/sonumaths2/