Department of Mathematics
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Minor Test I: August 29, 2011
MTL402: Algebra I(Hints to Solutions)

Duration: 1 hour M. Marks: 20

Note: Throughout, the symbol G will mean a Group.

1. (a) Let $|G| < \infty$; derive the class equation $|G| = |Z(G)| + \sum_{x \notin Z(G)} \frac{|G|}{|C_G(x)|}$. You may make use of action of G on it via conjugation. (4)

   Sol. Consider the action of G onto itself via conjugation i.e. $g \circ x = gxg^{-1}$, $\forall\, g, x \in G$. Under this action G can be written as a disjoint union of the orbits $\mathcal{O}_x := \{g \circ x \mid g \in G\}$ i.e. $G = \cup_{x \in G} \mathcal{O}_x = \cup_{x \in Z(G)} \mathcal{O}_x \cup \cup_{x \notin Z(G)} \mathcal{O}_x$. Note that if $x \in Z(G)$ then $\mathcal{O}_x = \{x\}$ since $gxg^{-1} = x$ for all $g \in G$. Therefore $|G| = \sum_{x \in Z(G)} |\mathcal{O}_x = \{x\}| + \sum_{x \notin Z(G)} |\mathcal{O}_x| = |Z(G)| + \sum_{x \notin Z(G)} |\mathcal{O}_x|$. We now prove that for all $x \in G$, $|\mathcal{O}_x| = |G|/|\mathrm{Stab}(x)|$. Here $\mathrm{Stab}(x) := \{g \in G \mid gxg^{-1} = x\} = C_G(x)$. We see that the surjective map $f : G/C_G(x) \to G$ defined by $f(gC_G(x)) = g \circ x = gxg^{-1}$ is injective sice $f(gC_G(x)) = f(hC_G(x)) \;\Rightarrow\; gxg^{-1} = hxh^{-1} \Rightarrow h^{-1}gx(h^{-1}g)^{-1} = x \;\Rightarrow\; h^{-1}g \in C_G(x) \Leftrightarrow gC_G(x) = hC_G(x)$. Hence $|G/C_G(x)| = |\mathcal{O}_x|$.

   Finally we prove that $|G/C_G(x)| = |G|/|C_G(x)|$; for this consider an action of G on the set $G/C_G(x)$ defined via $g \circ (xC_G(x)) = gxC_G(x)$. Under this action by the last result above we se that $|G| = |\mathcal{O}_{C_G(x)}||\mathrm{Stab}(C_G(x))| = |G/C_G(x)||C_G(x)|$.

   (b) Let $|G| < \infty$, $H \subset G$, and $G/H := \{gH \mid g \in G\}$. Show that map $\circ : G \times G/H \to G/H$ defined by $g \circ (xH) := (gx)H$ is a group action. Using it establish the following: $|G/H| = \frac{|G|}{|H|}$. (4)

   Sol. Since $g_1 \circ (g_2 \circ (xH)) = g_1 \circ (g_2 xH) = g_1(g_2 x)H = (g_1 g_2)xH$ for all $g_1, g_2 \in G$, $xH \in G/H$ where the associativity of G is used in the last step. Secondly $1_G \circ (xH) = 1_G xH = xH$, $\forall xH \in G/H$. Thus $\circ$ is an action of G on the set G/H. Under this action $\mathcal{O}_H = \{g \circ H \mid g \in G\} = \{gH \mid g \in G\} = G/H$ and $\mathrm{Stab}(H) = \{g \in G \mid g \circ H = H\} = \{g \in G \mid gH = H\} = \{g \in G \mid g \in H\} = H$. As $|G| = |\mathcal{O}_H||\mathrm{Stab}(H)| = |G/H||H|$ or $|G/H| = |G|/|H|$ as required.

   (c) Show that $|Z(Q_8)| = 2$, and $Z(S_3) = \{1\}$. (2)

   Sol. Note that the elements $\pm i, \pm j, \pm k$ of Q_8 are not in $Z(Q_8)$ because $ij = -ji \neq ji$, $jk = -kj \neq kj$, $ki = -ik \neq ik$ and since by definition $(\pm 1)x = \pm x = x(\pm 1)$ for all $x \in Q_8$, it follows that $Z(Q_8) = \{-1, 1\}$ or $|Z(Q_8)| = 2$.

   Now consider $S_3 := \{1_{S_3}, (12), (13), (23), (123), (132)\}$, and note that $(123) \circ (12) = (13) \neq (12) \circ (123) = (23)$; it follows that $(123), (12)$ and $(123)^{-1} = (132)$ are not in $Z(S_3)$. Similarly (23) and (13) do not commute with (123) and therefore are not members of $Z(S_3)$. Hence $Z(S_3) = \{1_{S_3}\}$.

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2. Prove the following (*attempt any five parts*):

(a) Quaternion group Q_8 is not isomorphic to dihedral group D_8. (2)

Sol. Q_8 has 6 elements $\pm i, \pm j, \pm k$ of order 4 but D_8 has only 2 elements r, r^3 of order four while to establish isomorphism between any two groups there should be same number of the elements of same order in both of them. Hence $Q_8 \not\cong D_8$.

(b) Let $|G| = n$. For any $x \in G$, $|x| \leq |G|$. (2)

Sol. Let $|x| = m$, then $1, x, x^2, ...x^{m-1}$ are all distinct elements of G for if $x^a = x^b, a > b$ for some $a, b \in \{1, 2, ..., m-1\}$ then $x^{a-b} = 1_G$ s.t. $0 < a - b < m$ which is a contradiction as $m = |x|$ is the least positive integer for which $x^m = 1_G$. Hence G has at least m elements i.e. $|x| \leq |G|$.

(c) If for $x \in G$ $|x| = n$ then for any integer k, $|x^k| = \dfrac{n}{\gcd(k, n)}$. (2)

Sol. Let $|x^k| = m$, and $d = \gcd(k, n)$,then m is least positive integer s.t. $(x^k)^m = x^{km} = 1$ since $|x| = n$ this means $n \leq km$ or $n/d \leq (k/d)m$ or $n/d \leq m$ since k/d is a positive integer. For reverse inequality note that $1 = x^n = (x^n)^{k/d} = (x^k)^{n/d}$ which shows that $m \leq n/d$. Hence $m = n/d$ as required.

(d) Let G, H be two groups and $\varphi : G \to H$ is a group ismorphism. Then $|x| = |\varphi(x)|$ for all $x \in G$. (2)

Sol. First we observe that $\varphi(1_G) = \varphi(1_G 1_G) = \varphi(1_G)\varphi(1_G) \Rightarrow \varphi(1_G) = 1_H$. Let $|x| = n$ then $x^n = 1_G$ and $1_H = \varphi(1_G) = \varphi(x^n) = \varphi(x)^n$; thus $|\varphi(x)| \leq n$. If we let $|\varphi(x)| = m$ then $\varphi(x)^m = 1_H$ or $\varphi(x^m) = 1_H = \varphi(1_G) \Leftrightarrow x^m = 1$ since φ is bijective. It follows that $n \leq m$. Result follows now.

(e) $H \subset G$ is a subgroup of G if and only if $H \neq \emptyset$ and $HH^{-1} \subset H$. (2)

Sol. If $H \leq G$ then H itself is a group under the operation of G. Therefor $H \neq \emptyset$ and $HH^{-1} \subset H$. Conversely let $H \neq \emptyset$ and $HH^{-1} \subset H$. Then there is a $x \in G$ such that $x \in H$. Then by hypothesis $1_G = xx^{-1} \in H$ and for each $x \in H$, $x^{-1} = 1_G x^{-1} \in H$. Since $H \subset G$ associative law holds good in H. Since H satisfies definition of a group, it follows that $H \leq G$.

(f) If $G = \langle x \rangle$ for some $x \in G$ then $|x| = |G|$. (2)

Sol. If $|G| = 1$ then nothing to prove so let $|G| > 1$ then $x \neq 1_G$. If $|G| = \infty$ then for any two integers $a, b, a \neq b$ $x^a \neq x^b$ otherwise if $x^a = x^b, a > b(say)$ then $x^{a-b} = 1$ but then $G =< x >= \{1, x, x^2, ..., x^{a-b}\}$ is a finite set a contradiction. Hence $x^a \neq x^b$ for all $a \neq b$. But then there is no positive integer n for which $x^n = 1_G$ hence $|x| = \infty = |G|$.

If $|G| = n > 1$. Then from Sol. Q2(b) it follows that $|x| \leq |G|$. It follows that $|G| \leq |x|$ since every element of G is of the form $x^t, t \in \mathbb{Z}$ and that there are only $|x|$ such different elements.

(g) Any two equivalence classes of an equivalence relation on a nonempty set are either the same or disjoint. (2)

Sol. Let $[a]$ and $[b]$ be two distinct equivalence classes of $\sim$. If possible let $x \in [a] \cap [b]$. Then $x \sim a$ and $x \sim b$ by symmetry of $\sim$, $a \sim x$ and via transitivity $a \sim x, x \sim b \Rightarrow a \sim b$ Thus $a \in [b]$. Now for every $y \in [a]$, $y \sim a$ and since $a \sim b$ applying transitivity again it follows that $y \sim b$ or $y \in [b]$. We have proved that $[a] \subset [b]$. Similarly by interchanging the role of a and b, we obtain that $[b] \subset [a]$.