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Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Minor Test II, 2011 for M.Sc.(Hons.)Mathematics, Semester-I

Algebra I | Max. Marks 20
MTL402 | Time allowed: 01 hour

Note: Throughout, the symbol G will mean a Group.

1. (a) Let $|G| < \infty$; and G has two composition series given by $1_G = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G$ and $1_G = N_0 \trianglelefteq N_1 \cdots \trianglelefteq N_{s-1} \trianglelefteq N_s = G$. Prove that $s = 2$ and the composition factors in these two series are isomorphic up to a permutation of $\{1, 2\}$. (6)

   (b) Let G be a finite group and p be a prime devisor of $|G|$. Prove that G has an element of order p. (3)

   (c) State and prove the second isomorphism theorem on groups. (3)

2. Prove the following (*attempt any four parts*):

   (a) Let G be finite group and $H \leq G$. Prove that $|H|$ divides $|G|$. (2)

   (b) Let $m,\ n$ be two positive integers each > 1 such that $|\bar{d}|$ divides $\gcd(m, n)$, $\bar{d} \in \mathbb{Z}/n\mathbb{Z}$. Prove that there is unique group homomorphism $\varphi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ such that $\varphi(\bar{1}) = \bar{d}$. (2)

   (c) Prove that $\mathrm{SL}_n(\mathbb{R}) \trianglelefteq \mathrm{GL}_n(\mathbb{R})$. For this you may look for a group homomorphism $\varphi : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ whose kernel is $\mathrm{SL}_n(\mathbb{R})$. (2)

   (d) Prove that every group of order p^2 for a prime p is abelian. (2)

   (e) Let $\varphi : G \to H$ be a group homomorphism and $E \leq H$. Prove that the inverse image $\varphi^{-1}(E) \leq G$. (2)

   (f) Write a presentation for the symmetric group $S_n, n \geq 3$. (2)