Roll No.——————

Major Test

M.Sc. Mathematics(Hons.) Semester-I

2013

Algebra I | Max. Marks 50

MTL402 | Max. time allowed: 03 hours

Note: Attempt all questions. The symbols G and 1_G stand for an abstract group and its identity element respectively. Also S_n and A_n denote the symmetric group and alternating groups respectively. Good Luck!

1. Answer the following in brief:
   (a) Let P be a Sylow-p-subgroup of G. If $P \trianglelefteq G$, then prove that $P \text{char} G$. (1)
   (b) Let G be a nonabelian group of order p^3($p-$prime). Prove that $[G, G] = Z(G)$. (1)
   (c) Construct a nonabelian group of order 24 with 7 elements of order 2. (1)
   (d) List up to isomorphism, all abelian groups of order 12. (1)
   (e) State Feit-Thompson theorem on solvability of finite groups. (1)

2. (a) Let G be a group and $H \trianglelefteq G$ and $K \trianglelefteq G$ such that $H \cap K = \{1_G\}$. Then prove that $HK \cong H \times K$ and $HK \trianglelefteq G$. (3)
   (b) Express the dihedral group $D_{4n},\ n = 1, 3, 5, ...,$ as a direct product of subgroups. (2)

3. (a) Prove that conjugate elements in the permutation group S_n have same cycle type. (3)
   (b) Prove that disjoint cycles commute in S_n. (2)

4. (a) If $H \leq S_n$ then prove that either $H \leq A_n$ or H has exactly $\frac{|H|}{2}$ even permutations. (3)
   (b) Let $H \leq G$. Establish that there is a homomorphism $\Psi : G \to S_{G/H}$ such that $\ker \Psi \leq H$ and for all $K \trianglelefteq G$ where $K \leq H$; then $K \leq \ker \Psi$. (2)

5. Let G be a finite group and p^α be a prime power that divides $|G|$. Prove that G has a subgroup of order p^α. (5)

6. Prove that the alternating group A_5 is simple. (5)

7. Prove that for all $\Gamma \in \text{Aut}(S_n),\ n = 3, 4, ...$ under the action of S_n on itself via conjugation
$$\mathcal{O}_{\Gamma((1\ 2))} = \begin{cases} \mathcal{O}_{((1\ 2))} & \text{if } n \neq 6 \\ \mathcal{O}_{((1\ 2))} \text{ or } \mathcal{O}_{((12)(34)(56))} & \text{if } n = 6 \end{cases}$$
where $\mathcal{O}_{(12)}$ is the orbit of (12). (5)

8. Let G be nonabelian group of order 28. Prove that either G is isomorphic to $Z_7 \rtimes Z_4$ or G is isomorphic to $Z_7 \rtimes (Z_2 \times Z_2)$. (5)

9. Let $|G| = pq,\ p < q$ where p, q are primes. Show that either $G \cong Z_{pq}$ or $G \cong Z_q \rtimes_\varphi Z_p$ where $\varphi : Z_p \to \text{Aut}(Z_q)$ is any nontrivial homomorphism. (5)

10. Let G be a finite group such that $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_r = G$ and $1 = M_0 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_s = G$ are two composition series for G. Then prove that $r = s$ and there is a permutation π of $\{1, 2, ..., r\}$ such that
$$\frac{M_{\pi(i)+1}}{M_{\pi(i)}} \cong \frac{N_{i+1}}{N_i}\ \forall\ i = 1, 2, ..., r-1.$$
You may assume that the result is true if $\min r, s = 2$. (5)