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Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Major Test, M.Sc. Mathematics(Hons.) Semester-I
2012

Paper: Algebra I **Max. Marks 50**

Code: MTL402 **Max. time allowed: 03 hours**

Note: Attempt all questions. The symbols G and 1_G stand for an abstract group and its identity element respectively. Also S_n and A_n denote the symmetric group and alternating groups respectively. Good Luck!

1. Classify upto isomorphism, *all* groups of order 66. (5)

2. If G is a finite group of order n and p is the smallest prime dividing n, prove that any subgroup of G of index p is normal in G. (5)

3. If $H \leq S_n$ then prove that either $H \leq A_n$ or H has exactly $\frac{|H|}{2}$ even permutations. (5)

4. Prove that the alternating group A_5 is simple. (5)

5. Prove that the alternating group A_n is simple for all $n \geq 5$. (5)

6. (a) Let $\sigma \in S_n$ such that $\sigma = \sigma_{n_1} \cdots \sigma_{n_k}$ be product of k disjoint cycles $\sigma_{n_i},\ i = 1, ..., k$ of lengths $n_1, ..., n_k$ respectively. Prove that $|\sigma| = \text{lcm}(n_1, ..., n_k)$. (3)

   (b) Let $m, n \in \mathbb{Z}^+ - \{1\}$. Prove that $Z_m \times Z_n \cong Z_{mn}$ if and only if $\gcd(m, n) = 1$. (2)

7. (a) Let G be a finite non-abelian simple group and $H \leq G$ such that $[G : H] = n$ for some positive integer n. Prove that G is isomorphic to a subgroup of A_n. (3)

   (b) Establish that a group of order 80 can not be simple. (2)

8. (a) Prove that a group G is solvable if and only if there is a positive integer n such that $G^{(n)} = \{1_G\}$ where $G^{(n)}$ is inductively defined as: $G^{(0)} = G$ and $G^{i+1} = [G^{(i)},\ G^{(i)}]$ for all $i = 0, 1, ..., n-1$. (3)

   (b) If $H \text{char} K \trianglelefteq G$ then prove that $H \trianglelefteq G$. (2)

9. (a) Prove that every finite abelian group G is isomorphic to a direct product of its Sylow subgroups. (3)

   (b) List up to isomorphism, all abelian groups of order 288. (2)

10. (a) Express the dihedral group $D_{2n},\ n = 3, 4, ...$ as a semidirect product. (3)

    (b) Show that $\text{Aut}(Z_{360})$ has 48 elements of order 12. (2)