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Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Supplementary Examination 2010 for M.Sc.(Maths) Semester I

Course name: Algebra I | **Max. Marks 100**
Course code: Math 514 | **Time Allowed: 03 Hours**

Note: Attempt any *ten* questions selecting two questions from each of the sections A, B, C, D, & E. Throughout the symbols G and 1_G denote an abstract group and its identity element respectively unless or otherwise specified.

Section A

1. Let $N \leq G$. Prove that $N \trianglelefteq G$ if and only if G/N is a group under the binary operation $*$ defined by $xN * yN = xyN$. (10)

2. Let p be a prime divisor of order of a finite group G. Prove that G has at least one subgroup of order p. (10)

3. (a) Let G be a finite group with even number of elements. Show that G has an element of order 2. (5)

   (b) Prove that for a finite group G, for every $x \in G$, $|x|$ divides $|G|$ and $x^{|G|} = 1_G$. (5)

4. (a) Prove that there is a one to one correspondence between the set of all left co sets of a subgroup $H \leq G$ in G and the sets of right co sets of H in G. (5)

   (b) Prove that the set of all transformations of the type $z \mapsto \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ of complex numbers to itself is a group over the composition of maps. (5)

Section B

5. Establish that (i) $\mathrm{Aut}(S_3) \cong S_3$ and (ii) $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$ where $n > 1$ is a positive integer and φ is the Euler phi function. (10)

6. If G is a finite group and p is the smallest prime dividing order of G then prove that any subgroup of G of index p is normal in G. (10)

7. (a) Let A, B be subgroups of G such that $A \leq N_G(B)$. Then prove that $B \trianglelefteq AB$ and $(A \cap B) \trianglelefteq A$. Prove further that $\dfrac{AB}{B} \cong \dfrac{A}{A \cap B}$. (5)

   (b) Let G' be the commutator subgroup of G. Show that $G' \trianglelefteq G$ and that the quotient group G/G' is abelian. (5)

8. (a) Prove that a subgroup of a cyclic group is always cyclic. (5)

   (b) Establish that every cyclic group of order 100 is isomorphic to the additive group $\mathbb{Z}/100\mathbb{Z}$ of residue classes modulo 100. (5)

Section C

9. Prove that any two elements of the permutation group S_n for a positive integer n are conjugate elements in S_n if and only if they have the same cycle type. In particular deduce that the total number of conjugates of a $m-$cycle in S_n for a positive integer $m \leq n$ is $m(n-m)!$ (10)

10. Prove that the alternating group A_5 is simple. (10)

11. (a) Let $G_1,\ G_2 \cdots, G_n$ be groups and $G = G_1 \times G_2 \times \cdots \times G_n$ be their direct product. Define for each $i = 1, 2, \cdots, n$ the projection map $\pi_i : G \to G_i$ by $\pi_i(g_1, g_2, ..., g_n) = g_i$. Prove that each π_i is a group homomorphism and that $G/\ker \pi_i \cong G_i$. Obtain the $\ker \pi_i$ explicitly. (5)

    (b) Write up to isomorphism all the abelian groups of order 180. (5)

12. (a) Prove that the center of a direct product of groups is the direct product of centers: $Z(G_1 \times G_2 \times \cdots \times G_n) \cong Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n)$. (5)

    (b) Deduce that a direct product of groups is abelian if and only if each of the factors is abelian. (5)

Section D

13. Prove that for a finite group G the number n_p of sylow-p-subgroups is equal to $|G|/|N_G(H)|$ where H is one of the Sylow-p-subgroups and that $n_p \equiv 1 \mod p$. (10)

14. Discuss the nature of Sylow-subgroups of a finite group G such that $|G| = p^2q$ where p and q are distinct primes. (10)

15. (a) Prove that every finite group possesses a composition series. (5)

    (b) Prove that every group of order 87 is solvable. (5)

16. (a) Derive the class equation for a finite group G. (5)

    (b) Let G be a group of order 143. Prove that G is cyclic. (5)

Section E

17. Let A be a subring and B be an ideal of a ring R. Then prove that $A+B := \{a+b \mid a \in A,\ b \in B\}$ is a subring of R, $A \cap B$ is an ideal of A and $\dfrac{A+B}{B} \cong \dfrac{A}{A \cap B}$. (10)

18. Prove that for a commutative ring R the ideal M of R is maximal if and only if the quotient ring R/M is a field. (10)

19. Construct the field of quotients of an integral domain. (10)

20. (a) Prove that a finite integral domain is a field. (5)

    (b) Let R be a ring with identity and S be a subring of R containing the identity. Prove that if u is a unit in S then u is a unit in R. Show by an example that the converse may not be always true. (5)