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Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Major Test 2011 for M.Sc.Maths(Hons.) Semester-I

MTL402 | **Max. Marks 50**
Algebra I (Group Theory) | **Time Allowed: 03 Hours**

Note: Symbols G and 1_G stand for an abstract group and its identity element respectively unless or otherwise specified. Students are encouraged to *be precise* in their attempt and avoid unnecessary arguments. Good luck!

1. Prove the following (be brief!):
   (a) $Z(S_n) \cong \{1\}$ for all $n \geq 3$ (1)
   (b) $G/Z(G) \cong \text{Inn}(G)$ (1)
   (c) If K char H and $H \trianglelefteq G$ then $K \trianglelefteq G$ (1)
   (d) G/G' is the largest abelian quotient of G in the sense that if $H \trianglelefteq G$ and G/H is abelian then $G' \subset H$. Here G' is the commutator subgroup of G. (1)
   (e) The symmetric group S_3 is solvable (1)

2. Let p be a prime number. Prove that the number n_p of Sylow-p subgroups with a representative subgroup H_p of a finite group G satisfies the following: (5)

$$n_p = \frac{|G|}{|N_G(H_p)|}; \; n_p \equiv 1 \mod p.$$

3. If $|G| = 30$ establish that G has a subgroup of order 15. (5)

4. If $|G| = pqr,\ p < q < r$ are primes then prove that G is not simple. (5)

5. Prove that a finite group G is solvable if and only if there exists a normal series for G whose factors are cyclic of prime orders. (5)

6. Assume that the alternating group A_5 is simple. Prove that the alternating group A_n is simple for all $n \geq 5$. (5)

7. Prove that every $\sigma \in S_n,\ n \geq 4, n \in \mathbb{Z}^+$ can be written as a product of disjoint cycles. (5)

8. (a) Prove Cayley's theorem: *every group is isomorphic to a subgroup of some symmetric group.* (2)
   (b) Let $|G| < \infty$ and p is the smallest prime dividing $|G|$. Then prove that any subgroup of G of index p is normal. (3)

9. (a) Prove that every finite abelian group G is isomorphic to a direct product of its Sylow subgroups. (3)
   (b) List up to isomorphism all abelian groups of order 24. (2)

10. (a) Suppose H and K are subgroups of G such that $H \trianglelefteq G$ and $H \cap K = \{1_G\}$. Consider the set $H \rtimes_\varphi K$ of all ordered pairs $(h, k),\ h \in H,\ k \in K$ by with binary operation $\rtimes$ defined by

$$(h_1, k_1) \rtimes (h_2, k_2) = (h_1 k_1 \circ h_2, k_1 k_2)$$

    where $\varphi : K \to \text{Aut}(H)$ is a group homomorphism which is defined by $\varphi(k)(h) = k \circ h = khk^{-1}$ for all $k \in K$ and $h \in H$. Show that $H \rtimes_\varphi K$ is a group. (3)
    (b) Show in *brief* that $S_3 \cong \mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_2$ under a group homomorphism $\varphi : \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z}_3)$. (2)