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# Department of Mathematics
## Guru Nanak Dev University, Amritsar-143005
### Major Test 2010 for M.Sc.Maths(Hons.) Semester-I

**Course name: Algebra I** — **Max. Marks 50**
**Course code: MTL402** — **Time Allowed: 03 Hours**

*Note:* The symbols $G$ and $1_G$ stand for an abstract group and its identity element respectively unless or otherwise specified.

1. (a) Let $|G| = 287$. Explain with an appropriate reasoning for whether there exists an element in $G$ of order 287 ?
   (b) If for $x \in G$, $|x| = n$ then establish that for any positive integer $a < n$, $|x^a| = \dfrac{n}{\gcd(a, n)}$.
   (c) If $\mathrm{Aut}(G) = \{1_G\}$ then deduce that $G$ is abelian and every non identity element of $G$ is of order two.
   (d) Establish that the dihedral group $D_{12}$ and the alternating group $A_4$ are not isomorphic.
   (e) Prove that a group of order 30 always has a subgroup of index two. $(5 \times 1)$

2. Prove the following:
   (a) The group table for a finite group is symmetric if and only if it is abelian.
   (b) $\mathrm{Aut}(S_n) \cong S_n$ for all $n \geq 3$.
   (c) Conjugate elements in $S_n$ have same cycle type.
   (d) For a nonabelian group $G$ of order $p^3$ where $p$ is a prime, $|Z(G)| = p$.
   (e) The group $\mathbb{Z}_3 \rtimes_\varphi \mathbb{Z}_4$ for a nontrivial homomorphism $\varphi : \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_3)$, is not isomorphic to either of the groups $A_4$ and $D_{12}$. $(5 \times 1)$

3. Let $G$ be a finite group. Prove that it has a composition series. Further if $G$ has two composition series given by $\{1_G\} = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G$ and $\{1_G\} = N_0 \trianglelefteq N_1 \trianglelefteq \cdots N_{r-1} \trianglelefteq N_r = G$ then prove that $r = 2$ and the composition factors in these composition series are isomorphic upto a permutation of $\{1, 2\}$. (5)

4. Prove that if a group of order 60 has more than one Sylow-5-subgroups then it is simple. (5)

5. Assume that the alternating group $A_5$ is simple. Prove that the alternating group $A_n$ is simple for all $n \geq 5$. (5)

6. Let $G$ be a finite group. If for a prime $p$ and a positive integer $\alpha$, $p^\alpha$ divides $|G|$, prove that $G$ has a subgroup of order $p^\alpha$. (5)

7. Let $|G| = pqr$ $(p < q < r)$ where $p,\ q,\ r$ are distinct primes. Discuss the simplicity of $G$ in various cases. Under what conditions $G$ is cyclic ? (5)

8. (a) Prove that every $p-$group is solvable. (3)
   (b) Prove that a group $G$ is solvable if and only if there is a positive integer $n$ such that $G^{(n)} = \{1_G\}$ where $G^{(n)}$ is inductively defined such that $G^{(0)} = G$ and $G^{i+1} = [G^{(i)},\ G^{(i)}]$ for all $i = 1, 2, ..., n-1$. (2)

9. (a) Prove that every finite abelian group $G$ is isomorphic to a direct product of its Sylow subgroups. (3)
   (b) List up to isomorphism all abelian groups of order 80. (2)

10. (a) Suppose $H$ and $K$ are subgroups of $G$ such that $H \trianglelefteq G$ and $H \cap K = \{1_G\}$ then prove that $HK \cong H \rtimes_\varphi K$ where $\varphi : K \to \mathrm{Aut}(H)$ is a group homomorphism which is defined by $\varphi(k)(h) = khk^{-1}$ for all $k \in K$ and $h \in H$. (3)
    (b) Construct a non-abelian group of order 27. (2)