Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Second Term Test: October 11, 2010
Algebra I
(Hints to Solutions)

Duration: 1 hour **M. Marks: 20**

Note: G denotes a group throughout

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1. **Prove that every finite group $G \neq \{1_G\}$ has a composition series.** **(4)**
If G is simple then the sequence of groups $1_G \trianglelefteq G$ is a composition series for G as the composition factor $G/1_G \cong G$ is simple. Suppose G is not simple then there is a nontrivial normal subgroup M of G. We prove the existence by induction on $|G|$. The result is already true for if $|G| = 2, 3$. Suppose all groups of orders $\leq m$ possess a composition series. Let $|G| = m+1$ then clearly $|M|$ and $|G/M|$ are less than $m+1$ and these groups possess composition series by induction hypothesis. Suppose the composition series are: $1 = M_0 \trianglelefteq M_1 \trianglelefteq \cdots M_{r-1} \trianglelefteq M_r = M$; $M = N_0/M \trianglelefteq N_1/M \trianglelefteq \cdots N_{s-1}/M \trianglelefteq N_s/M = G/M$; for some $r, s \in \mathbb{Z}^+$ such that composition factors M_{i+1}/M_i and $N_{j+1}/M \Big/ N_j/M \cong N_{j+1}/N_j$ (by third isomporphism theorem) are simple for all $1 \leq i \leq r-1$ and $1 \leq j \leq s-1$. Then the sequence of groups: $1 = M_0 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_{r-1} \trianglelefteq M_r = M \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{s-1} \trianglelefteq N_s = G$ is the required composition series for G. We have used the result $N_j/M \trianglelefteq N_{j+1}/M \Leftrightarrow N_j \trianglelefteq N_{j+1}$ and the 4th isomorphism theorem providing the one one one correspondence $N_j \leftrightarrow N_j/M$ for each j as above.

2. **For any finite group G, prove that $|G| = |\mathbf{Z}(G)| + \sum_{x \notin \mathbf{Z}(G)} \frac{|G|}{|C_G(x)|}$ where the summation is carried over the cardinalities of disjoint conjugacy classes.** **(4)**
Consider the decomposition of G into disjoint orbits under the action of G onto itself via conjugation i.e. $g \circ x = gxg^{-1}$ for all $g,\ x \in G$. Then the orbit $O_x := \{gxg^{-1} \mid g \in G\}$ and $Stab(x) = \{g \in G \mid gxg^{-1} = 1_G\} = C_G(x)$. Note that if $x \in Z(G)$ then $O_x = \{x\}$. Also the map $f : G/Stab(x) \to O_x$ defined by $gStab(x) \mapsto gxg^{-1}$ is well defined and clearly surjective therefore a bijection. Hence $|O_x| = |G/Stab(x)| = |G|/|C_G(x)|$. Hence $|G| = \sum_{x\in G} |G|/|C_G(x)| = \sum_{x \in Z(G)} |G|/|C_G(x)| + \sum_{x \notin Z(G)} |G|/|C_G(x)|$ where each equivalence class in the summation contributes exactly once. If $x \in Z(G)$ then $|G|/|C_G(x)| = 1$ and consequently $\sum_{x \in Z(G)} |G|/|C_G(x)| = \sum_{x\in Z(G)} 1 = |Z(G)|$. This proves the result.

3. **Let A, B be subgroups of G such that $A \leq N_G(B)$. Then prove that $B \trianglelefteq AB$ and $(A \cap B) \trianglelefteq A$. Prove further that $\frac{AB}{B} \cong \frac{A}{A \cap B}$.** **(4)**
Since $A \leq N_G(B)$ it follows that $aBa^{-1} = B$ for all $a \in A \Leftrightarrow AB = BA$ this proves that $AB \leq G$. $B \trianglelefteq AB$ because for all $a \in A,\ b \in B$ and $x \in B$, $(ab)x(ab)^{-1} = a\underbrace{bxb^{-1}}_{\in B}a^{-1} = B$ as $a \in N_G(B)$ which proves $B \trianglelefteq AB$. Also for all $a \in A$ and $y \in (A \cap B)$, $aya^{-1} \in A$ as A is a group and as $y \in B$ $aya^{-1} \in B$ hence $aya^{-1} \in (A \cap B)$. This establishes $(A \cap B) \trianglelefteq A$. Now we prove that the surjective map $\varphi : AB/B \to A/(A \cap B)$ defined by $\varphi(abB) = a(A \cap B)$ is a well defined map and is a homomorphism. This follows from $\varphi(a_1b_1Ba_2b_2B) = \varphi(a_1b_1a_2b_2B) = \varphi(\overbrace{a_1a_2}^{\in A}\overbrace{\underbrace{a_2^{-1}b_1a_2}_{\in B}b_2}^{\in B}B) = a_1a_2(A \cap B) = a_1(A \cap B)a_2(A \cap B) = \varphi(a_1b_1B)\varphi(a_2b_2B)$. If $\varphi(a_1b_1B) = \varphi(a_2b_2B)$ then $a_1a_2^{-1} \in (A \cap B)$ which means $a_1b_1b_2^{-1}a_1^{-1}a_1a_2^{-1} \in B \Leftrightarrow a_1b_1B = a_2b_2B$. This proves that φ is injective.

4. Prove the following:

(a) **A subgroup of G is normal in $G \Leftrightarrow$ it is equal to the kernel of some homomorphism.** **(2)**

Let $\varphi : G \to H$ be a group homomorphism such that a subgroup $H \leq G$ is $H = \ker \varphi$ then $\varphi(H) = 1_G$ and for all $g \in G$ $\varphi(gHg^{-1}) = \varphi(g)\varphi(H)\varphi(g)^{-1} = \varphi(g)\varphi(g)^{-1} = 1_H$. This means $gHg^{-1} \subset H$ hence $H \trianglelefteq G$. Conversely if $H \trianglelefteq G$ then it is equal to the $\ker f$ such that $f : G \to G/H$ defined by $f(g) = gH$ is a homomorphism. Because $f(g_1g_2) = g_1g_2H = g_1Hg_2H = f(g_1)f(g_2)$ and $\ker \varphi = \{g \in G \mid gH = 1_H\} = \{g \in G \mid g \in H\} = H$.

(b) **$\Psi : \mathbf{Aut}(\mathbb{Z}_n) \to (\mathbb{Z}/n\mathbb{Z})^\times$ such that $\Psi(\psi_a) = a \pmod{n}$ is a group isomorphism where $\psi_a : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $\psi_a(x) = x^a$ is an automorphism of $\mathbb{Z}_n$.** **(2)**

Let $a, b \in \mathbb{Z}_n$ then for all $x \in \psi_a \circ \psi_b(x) = \psi_a(\psi_b(x)) = \psi_a(x^b) = (x^b)^a = x^{ab} = \psi_{ab}(x)$. Therefore $\psi_a \circ \psi_b = \psi_{ab}$ and $\Psi(\psi_a \circ \psi_b) = \Psi(\psi_{ab}) = ab(\text{mod} n) = a(\text{mod} n)b(\text{mod} n) = \Psi(\psi_a)\Psi(\psi_b)$. This proves that Ψ is a homomorphism. Clearly Ψ is surjective and to establish that it is injective let $\Psi(\psi_a) = \Psi(\psi_b)$ then $\Psi(\psi_a\psi_b^{-1}) = \Psi(\psi_{ab^{-1}}) = 1(\text{mod} n)$ which gives $ab^{-1} = 1(\text{mod} n)$ or $a = b(\text{mod} n)$. Note that we have used $\psi_{aa^{-1}}(x) = x = \psi_{a^{-1}a}(x)$ for all x i.e. $\psi_a^{-1} = \psi_{a^{-1}}$.

(c) **If $H \leq G$ such that $|G : H| = 2$ then $H \trianglelefteq G$.** **(2)**

First note that the map $gH \mapsto Hg^{-1}$ is a bijection between the set of left cosets of H in G and the set of right cosets of H in G. Since $|G : H| = 2$ the set of left cosets of Hin G has exactly two elements i.e. H and xH for $x \notin H$. Then the set of right cosets of H in G also has two elements H and Hx. Since $H \cap xH = \emptyset = H \cap Hx$ and $G = H \cup xH = H \cup Hx$ it follows that $xH = Hx \Rightarrow H \trianglelefteq G$.

(d) **Every element of the additive quotient group $\mathbb{Q}/\mathbb{Z}$ is of finite order.** **(2)**

Any element of $\mathbb{Q}/\mathbb{Z}$ is of the form $\dfrac{p}{q} + \mathbb{Z} := \{p/q + z \mid q \neq 0,\ p,\ q,\ z \in \mathbb{Z}\}$. Then $|q|\left(\dfrac{p}{q} + \mathbb{Z}\right) = \{|q|p/q + z|q| \mid q \neq 0,\ p,\ q,\ z \in \mathbb{Z}\} = \{\ \text{sign}(q)p + z|q| \mid q \neq 0,\ p,\ q,\ z \in \mathbb{Z}\} = \mathbb{Z}$ which the identity element of $\mathbb{Q}/\mathbb{Z}$. Since $|q|$ is always finite each element of this group is of finite order.