Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Second Term Test: October 11, 2010
Algebra I

Duration: 1 hour — M. Marks: 20
Note: G denotes a group throughout

1. Prove that every finite group $G \neq \{1_G\}$ has a composition series. (4)
2. For any finite group G, prove that $|G| = |\mathrm{Z}(G)| + \sum_{x \notin \mathrm{Z}(G)} \frac{|G|}{|C_G(x)|}$ where the summation is carried over the cardinalities of disjoint conjugacy classes. (4)
3. Let $A,\ B$ be subgroups of G such that $A \leq N_G(B)$. Then prove that $B \trianglelefteq AB$ and $(A \cap B) \trianglelefteq A$. Prove further that $\frac{AB}{B} \cong \frac{A}{A \cap B}$. (4)
4. Prove the following:
   (a) A subgroup of G is normal in $G \Leftrightarrow$ it is equal to the kernel of some homomorphism.
   (b) $\Psi : \mathrm{Aut}(\mathbb{Z}_n) \to (\mathbb{Z}/n\mathbb{Z})^{\times}$ such that $\Psi(\psi_a) = a \pmod n$ is a group isomorphism where $\psi_a : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $\psi_a(x) = x^a$ is an automorphism of $\mathbb{Z}_n$.
   (c) If $H \leq G$ such that $|G : H| = 2$ then $H \trianglelefteq G$.
   (d) Every element of the additive quotient group $\mathbb{Q}/\mathbb{Z}$ is of finite order. (4×2)

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