Department of Mathematics

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Algebra I

Note: All question carry equal marks Duration: 1 hour

1. Let G be a group and $M \leq G$ such that we define two sets $xM := \{xm \mid m \in M\}$ and $\mathcal{M} := \{xM \mid x \in G\}$. Prove that the map $\circ : G \times \mathcal{M} \to \mathcal{M}$ defined by $g \circ xM := (gx)M$ is an action of G onto $\mathcal{M}$. In particular if $G = S_3$ and $M = \{1, (12)\}$, obtain the orbit O_M and the stabilizer Stab(M) of M under this action and verify that
$$|S_3| = |O_M||\text{Stab}(M)|.$$
2. Let m and n be two positive integers such that $mr + ns = 1$ for some integers r and s. Suppose $x^t y^t = y^t x^t$, $t = m, n$ for all $x, y \in G$. Then show that $xy = x^{mr}y^{-mr}(yx)y^{mr}x^{-mr}$ for all $x, y \in G$. Deduce that G is abelian.
3. Prove that a cyclic group is isomorphic either to $(\mathbb{Z}_n, +)$ for some positive integer n or to $(\mathbb{Z}, +)$.
4. Establish the following (i) $(\mathbb{Q}, +) \ncong (\mathbb{Q}^+, \times)$ (ii) $(\mathbb{R}, +) \cong (\mathbb{R}^+, \times)$.
5. Prove that the Klein-4 group $\mathbb{D}_4$ is not isomorphic to the additive group of residue classes $\mathbb{Z}_4$ modulo 4.

Solutions

1. **Sol.** For $g_1, g_2, x \in G, xM \in \mathcal{M}$ we have $g_1 \circ (g_2 \circ xM) := g_1 \circ (g_2 xM) = (g_1(g_2 x))M = g_1 g_2(xM) = (g_1 g_2) \circ xM$. Also $1_G \circ xM = 1_G xM = xM$ for all $xM \in \mathcal{M}$. This establishes that $\circ : G \times \mathcal{M} \to \mathcal{M}$ is a group action. Under this action $M = \{1, (12)\} \leq S_3$ and $M \in \mathcal{M}$. Therefore $O_M = \{\sigma \circ M \mid \sigma \in S_3\} = \{\sigma, \sigma(12) \mid \sigma \in S_3\} = S_3$. Also $Stab(M) = \{\sigma \in S_3 \mid \sigma \circ M = M\} = \{(1)\}$. Clearly $|O_M| = |S_3| = 3! = 6$ and $|Stab(M)| = 1$ which verifies $|S_3| = |O_M||Stab(M)|$.
2. **Sol.** $xy = x^{mr+ns}y^{mr+ns} = x^{mr}(x^s)^n(y^s)^n y^{mr} = x^{mr}(y^s)^n(x^s)^n y^{mr} = x^{mr}y^{1-mr}x^{1-mr}y^{mr} = x^{mr}y^{-mr}yxx^{-mr}y^{mr} = x^{mr}y^{-mr}(yx)y^{mr}x^{-mr}$ as required where we have used $ns = 1 - mr$ and mth powers (similarly nth powers) of elements of G commute.
Next to prove that G is abelian first observe that for any positive integer k
$$(xy)^k = \underbrace{x^{mr}y^{-mr}(yx)y^{mr}x^{-mr} \ \cdots \ x^{mr}y^{-mr}(yx)y^{mr}x^{-mr}}_{k-\text{times}} = x^{mr}y^{-mr}(yx)^k y^{mr}x^{-mr}.$$
A similar relation holds if we replace m by n. In particular take $k = mr, ns$ to establish $(xy)^{mr} = x^{mr}y^{-mr}(yx)^{mr}y^{mr}x^{-mr} = (yx)^{mr}$ and $(xy)^{ns} = (yx)^{ns}$ which directly gives us $(xy)^{mr+ns} = (yx)^{mr+ns}$ that is $xy = yx$.
3. **Sol.** Let $G =< x >$. If $|G| = \infty$ then the surjective map $f : \mathbb{Z} \to G$ defined by $n \mapsto x^n$ is an isomorphism since $f(n_1 + n_2) = x^{n_1+n_2} = x^{n_1}x^{n_2} = f(n_1)f(n_2)$ which proves that f is a homomorphism. Also for any two nonidentity elements x^s and x^t of G $x^t \neq x^s$ whenever $s \neq t$ as each of such elements of G are of infinite order. This proves that f is injective. Hence in this case $G \cong \mathbb{Z}$.
The other part when $|G| = n$ is finite can be proved in exactly same manner as above. Here the isomorphism is $\phi : \mathbb{Z}_n \to G$ by $\bar{m} \mapsto x^m$.
4. **Sol.** (i) To prove $(\mathbb{Q}, +) \ncong (\mathbb{Q}^+, \times)$ suppose if possible suppose $f : (\mathbb{Q}, +) \to (\mathbb{Q}^+, \times)$ be an isomorphism. Then as f is a surjective homomorphism for $2 \in \mathbb{Q}^+$ there is $x \in \mathbb{Q}$, such that $2 = f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{x}{2}\right)^2$ which means $f\left(\frac{x}{2}\right) = \sqrt{2} \Rightarrow\Leftarrow$.
(ii) $(\mathbb{R}, +) \cong (\mathbb{R}^+, \times)$ as the map $(\mathbb{R}, +) \to \mathbb{R}^\times$ defined by $x \mapsto e^x$ is an isomorphism.
5. **Sol.** Klein-4 group $\mathbb{D}_4$ is not isomorphic to the additive group of residue classes $\mathbb{Z}_4$ modulo 4 because $\bar{1} \in \mathbb{Z}_4$ is of order 4 and there is no element of order 4 in the Klein-4 group. Because under an isomorphism, order of any group element and its image are always same.

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