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Notes on 'Jordan Holder's Theorem' & assignment #3

Abstract

In these notes we introduce sequences of groups and prove a structure theorem on groups namely 'the Jordan-Holder's theorem'

Definition: A group is called simple if it does not have any normal subgroup other than 1_G and G.

For example Z_p for $p-$prime is simple. Later on we will prove that $A_n,\ n \geq 5$ is also simple. D_{2n} is not simple because $R_{2n} \trianglelefteq D_{2n}$.

Lemma 0.1. *Let $\varphi : G \to \varphi(G)$ be a homomorphism. If G is simple then so is $\varphi(G)$.*

Proof. Let G be simple and $K \trianglelefteq \varphi(G)$. We first prove that the inverse image $\varphi^{-1}(K) \leq G$. To establish this first note that $1_G \in \varphi^{-1}(K)$ because $\varphi(1_G) = 1_{\varphi(G)} = 1_K$. Therefore $\varphi^{-1}(K) \neq \emptyset$. Let $x, y \in \varphi^{-1}(K)$ then $\varphi(x), \varphi(y) \in K$ and since $K \leq \varphi(G)$ it follows that $\varphi(x)\varphi(y)^{-1} \in K$ or $\varphi(xy^{-1}) \in K$ this implies $xy^{-1} \in \varphi^{-1}(K)$. Hence $\varphi^{-1}(K) \leq G$. Now we show that $\varphi^{-1}(K) \trianglelefteq G$ to see this, consider for all $x \in \varphi^{-1}(K)$ and $y \in G$,

$$\varphi(yxy^{-1}) = \varphi(y)\varphi(x)\varphi(y)^{-1} \in K$$

since $K \trianglelefteq \varphi(G)$. It follows that $yxy^{-1} \in \varphi^{-1}(K)$. Thus $\varphi^{-1}(K) \trianglelefteq G$. Since G is simple, it follows that either $\varphi^{-1}(K) = 1_G$ or $\varphi^{-1}(K) = G$ i.e. either $K = \varphi(1_G)$ or $K = \varphi(G)$ which proves that $\varphi(G)$ is simple. □

Definition: (Composition series) Let G be a group. A sequence of subgroups $1 = N_0 \leq N_1 \leq \cdots \leq N_{k-1} \leq N_k = G$ is called a composition series if $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i is simple for all $i = 1, 2, ..., k-1$. The quotient groups N_{i+1}/N_i are called composition factors.

Example: The following two sequences of subgroups of the dihedral group D_8

$$1 \trianglelefteq \langle s \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8;\ 1 \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle r \rangle \trianglelefteq D_8$$

are two composition series for D_8, each of the three composition factors in each of the sequences being isomorphic to Z_2 is simple.

The following lemma will be used in the proof of Jordan-Hölder's theorem.

Lemma 0.2. *Let $H \leq M_1 \leq M_2 \leq G$ be a sequence of subgroups of a group G such that $H \trianglelefteq M_1$ and $H \trianglelefteq M_2$. Then*

$$M_1 \trianglelefteq M_2 \Leftrightarrow M_1/H \trianglelefteq M_2/H.$$

Proof. Since H is normal subgroup of M_1 and M_2, we see that $M_1/H \trianglelefteq M_2/H \Leftrightarrow$ for all $m_1 \in M_1$, $m_2 \in M_2$,

$$\begin{aligned} &m_2H(m_1H)(m_2H)^{-1} \in M_1/H \\ &\Leftrightarrow (m_2m_1m_2^{-1})H \in M_1/H \\ &\Leftrightarrow m_2m_1m_2^{-1} \in M_1 \Leftrightarrow M_1 \trianglelefteq M_2. \end{aligned}$$

This completes the proof. □

Lemma 0.3. *Suppose $1 \trianglelefteq M \trianglelefteq G$ and $1 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = G$ are two composition series of a finite group G then $k = 2$ and the composition factors are isomorphic up to a permutation of $\{1, 2\}$.*

Proof. First note that $k \geq 2$ because if $k < 2$ then G will be simple and nothing is left to prove. Given that $1 \trianglelefteq M \trianglelefteq G$ is a composition series, therefore the composition factors $M/1$ and G/M are simple. Consider the sequence of subgroups

$$\begin{aligned} 1 \leq (M \cap N_1) \leq \cdots \leq (M \cap N_{k-1}) \leq \\ \leq (M \cap N_k) = M \leq MN_1 \leq \cdots \leq MNk = G. \end{aligned} \tag{0.1}$$

Claim I: $\forall i = 1, ..., k-1$, $(M \cap N_i) \trianglelefteq (M \cap N_{i+1})$. This follows since for all $x \in (M \cap N_{i+1})$, $m \in M$, $xmx^{-1} \in M$ since $x \in M$ and $xN_ix^{-1} = N$ since $x \in N_{i+1}$ and $N_i \trianglelefteq N_{i+1}$. Hence $x(M \cap N_i)x^{-1} = M \cap N_i$. This proves that claim.

Claim II: $MN_i \trianglelefteq MNi + 1$.
Note that $MN_i \leq G$ since $M \trianglelefteq G$. Let $mn \in MN_{i+1}$ then

$$mnMMN_in^{-1}m^{-1} = m \underbrace{nMn^{-1}}_{\in M \because M \trianglelefteq G} \overbrace{nN_in^{-1}}^{\in N_i \because N_i \trianglelefteq N_{i+1}} m^{-1}$$

$$\subseteq mMN_im^{-1} = MN_im^{-1} = N_iMm^{-1} = N_iM = MN_i$$

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since $m \in M$ and $MN_i \leq G$. The claim follows now. Since $M/1 (\cong M)$ is simple therefore M is simple. Therefore from Eq.(0.1) it follows that either $M \cap N_{k-1} = 1$ or $M \cap N_{k-1} = M$. We have two cases.

Case I: $M \cap N_{k-1} = 1$. If $M \cap N_{k-1} = 1$ then $M \cap N_i = 1, \ \forall i = 1, ..., k-1$. Then $|MN_i| = |M||N_i|$ and since $1 \lneq N_1 \lneq \cdots \lneq N_{k-1}$ it follows that

$$|M| < |MN_1| < \cdots < |MN_{k-1}|.$$

Therefore $1 = M/M \lneq MN_1/M \lneq \cdots \lneq G/M$ where G/M is simple! This is possible only if $k = 2$. Hence $1 \trianglelefteq N_1 \trianglelefteq G$ and $MN_1 = G$. Consequently

$$G/M \cong MN_1/M \cong N_1/(N_1 \cap M) \cong N_1,$$

$$M \cong M/(M \cap N_1) \cong MN_1/N_1 \cong G/N_1$$

where we have used 2nd isomorphism theorem such that

$$MN_1/M \cong N_1/(N_1 \cap M), \ MN_1/N_1 \cong M/(N_1 \cap M).$$

Therefore in this case the composition factors in the composition series $1 \trianglelefteq M \trianglelefteq G$ and $1 \trianglelefteq N_1 \trianglelefteq G$ are isomorphic up to the permutation (12) of $\{1, 2\}$.

Case II: $M \cap N_{k-1} = M$. In this case $M \leq N_i$ for all $i = 1, ..., k-1$ and $MN_i = N_i$. From Eq.(0.1) we obtain $M \lneq N_1 \lneq \cdots \lneq N_{k-1} \lneq G$ with G/M simple. Therefore again $k = 2$ and $1 \trianglelefteq N_1 \trianglelefteq G$. In this case $M = N_1$ and the two composition series for G are exactly same. So here the composition factors are isomorphic upto the identity permutation of $\{1, 2\}$. □

Theorem 0.4. (Jordan-Hölder theorem)

1. *Every finite group $G \neq 1_G$ has a composition series.*
2. *The composition factors in a composition series are unique in the following sense: if $1 = N_0 \leq N_1 \leq \cdots N_r = G$ and $1 = M_0 \leq M_1 \leq \cdots \leq M_s = G$ are two composition series for G, then $r = s$ and there is a permutation π of $\{1, 2, ..., r\}$ such that*

$$M_{\pi(i)+1}/M_{\pi(i)} \cong N_{i+i}/N_i \ , \forall \ i = 1, 2, ..., r-1.$$

Proof. Part 1. We will prove the existence of composition series for G via induction on $|G|$. If $|G| = 2$ then $1_G \trianglelefteq G$ is a composition series. Suppose all groups of order $\leq m-1$ have a composition series. Now let $|G| = m$. If G is simple then $1_G \trianglelefteq G$ is composition series for G otherwise there is a proper normal subgroup H of G such that $H \neq 1_G, G$. Then $|H| < |G| = m$ and also $|G/H| < m$. By induction hypothesis, H and G/H have composition series. Let these be the following:

$$1 = H_0 \leq H_1 \leq \cdots \leq H_r = H,$$

$$1 = M_0/H \leq M_1/H \leq \cdots \leq M_s/H = G/H$$

where M_i's are subgroups of G each of which contain H. From lemma 0.2 it follows that

$$M_i/H \trianglelefteq M_{i+1}/H \Leftrightarrow M_i \trianglelefteq M_{i+1} \ \forall \ i = 1, ..., r-1.$$

Using third isomorphism theorem it follows that

$$(M_{i+1}/H)/(M_i/H) \cong M_{i+1}/M_i, \ \forall \ i = 1, ..., r-1$$

where the composition factors $(M_{i+1}/H)/(M_i/H)$ are simple, therefore using lemma 0.1 the composition factors M_{i+1}/M_i are also simple. It follows that the Sequence of subgroups given by

$$1 = H_0 \trianglelefteq \cdots \trianglelefteq H_r = H \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_s = G$$

is a composition series for G. This completes the final step of induction.

Part 2. Let

$$1 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_r = G$$

and

$$1 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_s = G$$

be two composition series for the finite group G. We will prove the result via induction on $k = \min\{r, s\}$ with arbitrary G. The result is true for $k = 2$ from the last lemma 0.3. Suppose that the result holds for any G and any composition series for which $\min r, s \leq k-1$. Let us suppose that $\min r, s = k$. W.l.o.g. we can take $k = r$. Then $M_{r-1} \trianglelefteq G$ and G/M_{r-1} is simple. We have the following

$$1 \trianglelefteq M_{r-1} \cap N_1 \trianglelefteq \cdots \trianglelefteq M_{r-1} \cap N_{s-1} \qquad (0.2)$$

and

$$1 \trianglelefteq M_1 \cap N_{s-1} \trianglelefteq \cdots \trianglelefteq M_{r-1} \cap N_{s-1} \qquad (0.3)$$

are two composition series for $M_{r-1} \cap N_{s-1}$ with $s-1$ and $r-1$ factors respectively since the composition factors

$$\frac{M_{r-1} \cap N_{i+1}}{M_{r-1} \cap N_i} \cong \frac{M_{r-1} \cap N_{i+1}}{(M_{r-1} \cap N_{i+1}) \cap N_i} \cong \frac{(M_{r-1} \cap N_{i+1})N_i}{N_i}.$$

where

$$\frac{(M_{r-1} \cap N_{i+1})N_i}{N_i} \trianglelefteq \frac{N_{i+1}}{N_i}.$$

Since $\frac{N_{i+1}}{N_i}$ is simple therefore

$$\frac{M_{r-1} \cap N_{i+1}}{M_{r-1} \cap N_i} \cong \left(1 \ \text{ or } \ \frac{N_{i+1}}{N_i}\right).$$

In either of the cases the factor is simple. By induction hypothesis the result is true for composition series with

$r-1$ and $s-1$ factors therefore $r-1=s-1$ which gives $r=s$ and the composition factors in the two composition series given by Eq.(0.2) and Eq.(0.3) are isomorphic upto a permutation of $\{1,...,r-1\}$. Now we have two composition series for M_{r-1} namely

$$1 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_{r-1}$$

and

$$1 \trianglelefteq M_{r-1}\cap N_1 \trianglelefteq \cdots \trianglelefteq M_{r-1}\cap N_{r-1} \trianglelefteq M_{r-1}$$

with $r-1$ and r composition factors respectively. Since both the composition series must contain equal number of factors, it follows that there are exactly two terms same in the latter composition series for M_{r-1}. Let $M_{r-1}\cap N_i = M_{r-1}\cap N_{i+1}$ for some $i \in \{1,2,...,r-2\}$. Since by induction hypothesis the composition factors in the above two composition series are isomorphic upto a permutation σ of $\{1,2,...,r-1\}$ i.e. for all $j=0,...,r-2$,

$$\frac{M_{r-1}\cap N_{j+1}}{M_{r-1}\cap N_j} \cong \begin{cases} \dfrac{M_{\sigma(j)+1}}{M_{\sigma(j)}} & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases} \tag{0.4}$$

and the only remaining un-compared factors in the two series are $\dfrac{M_{r-1}}{M_{r-1}\cap N_{r-1}}$ and $\dfrac{M_{\sigma(i)+1}}{M_{\sigma(i)}}$ respectively, therefore they must also be isomorphic i.e.

$$\frac{M_{r-1}}{M_{r-1}\cap N_{r-1}} \cong \frac{M_{\sigma(i)+1}}{M_{\sigma(i)}}. \tag{0.5}$$

We have already seen that for all $j=1,...,r-2$

$$\frac{M_{r-1}\cap N_{j+1}}{M_{r-1}\cap N_j} \cong \begin{cases} \frac{N_{j+1}}{N_j} \text{ if } j \neq i \\ 1 \text{ if } j = i. \end{cases} \tag{0.6}$$

and here again the two un-compared factors are left in the two series which must be isomorphic i.e.

$$\frac{M_{r-1}}{M_{r-1}\cap N_{r-1}} \cong \frac{M_{r-1}N_{r-1}}{N_{r-1}}$$

Note that since $M_{r-1} \trianglelefteq G$ and $N_{r-1} \trianglelefteq G$, it follows that $M_{r-1} \leq G = N_G(N_{r-1})$ therefore

$$\frac{M_{r-1}N_{r-1}}{N_{r-1}} \trianglelefteq \frac{G}{N_{r-1}}.$$

Since the factor $\frac{G}{N_{r-1}}$ is simple and $\frac{M_{r-1}N_{r-1}}{N_{r-1}} \neq 1 (\because r > 2)$ therefore

$$\frac{M_{r-1}N_{r-1}}{N_{r-1}} = \frac{G}{N_{r-1}}.$$

We have proved that

$$\frac{M_{\sigma(j)+1}}{M_{\sigma(j)}} \cong \begin{cases} \dfrac{N_{j+1}}{N_j} \text{ if } j \neq i \\ \dfrac{N_r}{N_{r-1}} \text{ if } j = i \end{cases}, \forall j = 0,...,r-2 \tag{0.7}$$

where we extend σ to a permutation of S_r. Similarly now interchanging the role of M_{r-1} and N_{r-1} the composition factor M_r/M_{r-1} must be isomorphic to one of the factors among $\frac{N_{j+1}}{N_j}$ for some j. The only left over un compared factor in the latter list is $\frac{N_{i+1}}{N_i}$. Thus

$$\frac{G}{M_{r-1}} \equiv \frac{N_{i+1}}{N_i}.$$

Finally define $\pi \in S_r$ such that

$$\pi(j) = \begin{cases} \sigma(j) & \text{if } j \neq i, r-1 \\ r-1 & \text{if } j = i \\ i & \text{if } j = r-1 \end{cases}$$

then

$$\frac{M_{\pi(j)+1}}{M_{\pi(j)}} \cong \frac{N_{j+1}}{N_j} \ \forall\ j = 0,...,(r-1).$$

This completes the proof. □

A partial converse of part 1 of Jordan-Hölder theorem is also true for *abelian groups*. Let us state and prove it now.

Theorem 0.5. *If G is abelian group such that G has a composition series then G is finite.*

Proof. We first show that if a group K is simple-abelian group then K can not be an infinite group. If possible let K be of infinite order. If K is not-cyclic then there is a $x \neq 1_K$ s.t. $\langle x \rangle$ is a proper normal subgroup of K which contradicts the fact that K is simple. Therefore K must be cyclic. Since every infinite cyclic group is isomorphic to $\mathbb{Z}$ which is not simple, it follows that K must be a finite-simple-cyclic group. But an abelian-cyclic group is simple if and only if its order is a prime number (otherwise Cauchy's theorem will enable its proper normal subgroup).
Now we prove the theorem. Let

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$$

be a composition series for abelian group G. Since each of the composition factors $H_{i+1}/H_i,\ i=0,...,r-1$ are simple-abelian, therefore by the above discussion

$$|H_{i+1}/H_i| = p_i,\ \forall i = 0,...,r-1$$

for some prime p_i. Observe that each H_i is finite! Then

$$p_0 \cdots p_{r-1} = \frac{|H_1|}{|H_0|}\frac{|H_2|}{|H_1|}\cdots\frac{|H_r|}{|H_{r-1}|}$$

which gives $|G| = |H_r| = p_0 \cdots p_{r-1}$ which is finite. □

We will not use Jordan-Hölder theorem in proving any other result here except later when we will study solvable groups. We pause here and prove the following version of the Cauchy's theorem.

Theorem 0.6. (Cauchy's theorem) *Let G be a finite group and p is a prime divisor of $|G|$. Then G has an element of order p.*

Proof. Define the set

$$S = \{(x_1, ..., x_p) \mid x_i \in G, \ \& \ x_1 \cdots x_p = 1\}.$$

Then a $p-$tuple $(x_1, ..., x_p) \in S$ if and only if each $x_i \in G$ and $x_p = (x_1 \cdots x_{p-1})^{-1}$. This enables us easy calculations of $|S|$. Clearly each of the first $p-1$ components in any element of S can be selected from any of the members of G and the last $p-$the component is fixed, therefore

$$|S| = \underbrace{|G| \times \cdots \times |G|}_{(p-1)-\text{times}} \times 1 = |G|^{p-1}.$$

Let $H = \langle (12...p) \rangle$ is the subgroup of S_p consisting of all cyclic permutations of $\{1, ..., p\}$. Then under the action of H on S define by

$$\sigma \bullet (x_1, ..., x_p) = (x_{\sigma(1)}, ..., x_{\sigma(p)})$$

we obtain a partition of S consisting of all orbits in S. For each $x = (x_1, ..., x_p) \in S$,

$$\mathcal{O}_x := \{\sigma \bullet x \mid \sigma \in H\}.$$

Note that $|\mathcal{O}_x| = 1$ if and only if $\sigma \bullet x = x \forall \sigma \in H \Leftrightarrow x_1 = x_{\sigma(1)} = x_{\sigma^2(1)} \cdots = x_{\sigma^{p-1}(1)}$ such that $1 \neq \sigma(1) \neq \sigma^2(1) \cdots \neq \sigma^{p-1}(1)$. It follows that $x = (a, ..., a)$ for some $a \in G$. If $|\mathcal{O}_x| > 1$ then clearly $|\mathcal{O}_x| = p$ since $|\text{Stab}(x)| = 1$. Therefore

$$|S| = |G|^{p-1} = k + pd$$

where k is the number of singleton orbits and d is the number of orbits with $p-$elements each. Clearly p divides k which shows that $k \geq p \geq 2$ and that there is a singleton orbit other than $\{(1, ..., 1)\}$ let it be $\{(a, ..., a)\}$ for some $a \in G$ s.t. $a \neq 1_G$, then $a^p = 1$ and hence $|a| = p$. $\square$

We now give two applications of the class equation.

Proposition 0.7. *If $|G| = p^\alpha$, $\alpha \geq 1$ for some prime number p, then $Z(G) \neq \{1_G\}$.*

Proof. From the class equation we have

$$|Z(G)| = p^\alpha - \sum_{x \notin Z(G)} \frac{p^\alpha}{|C_G(x)|}.$$

Since $1 < |C_G(x)| < |G| = p^\alpha$, it follows p divides $\frac{p^\alpha}{|C_G(x)|}$ for all $x \notin Z(G)$. Hence p divides $p^\alpha - \sum_{x \notin Z(G)} \frac{p^\alpha}{|C_G(x)|} = |Z(G)|$. $\square$

Corollary 0.8. *Every group of order p^2 (p-prime) is abelian.*

Proof. Let $|G| = p^2$. Then by proposition 0.7 $|Z(G)| \neq 1$. Since $|Z(G)|$ divides p^2, therefore either $|Z(G)| = p$ or $|Z(G)| = p^2$. If $|Z(G)| = p^2$ then $Z(G) = G$ and hence G is abelian. If $|Z(G)| = p$ then $|G/Z(G)| = p$ which means $G/Z(G)$ is cyclic. Let $G/Z(G) = \langle xZ(G) \rangle$. Then $x^p \in Z(G)$. If $y, z \in G$ then $yZ(G) = x^m Z(G)$ and $zZ(G) = x^n Z(G)$ for some integers m and n. Then $yx^m, zx^n \in Z(G)$. There exist $t, s \in Z(G)$ such that $yx^m = s$ and $zx^n = t$. Then

$$yz = sx^{-m}tx^{-n} = stx^{-m-n} = tx^{-n}sx^{-m} = zy$$

where we have used $su = us, tu = ut$ for all $u \in G$. Hence in this case also G is abelian. $\square$

Every group of order p^2 for a prime p is either isomorphic to Z_{p^2} or $Z_p \times Z_p$ as can be seen from the following. If G is cyclic then $G \cong Z_{p^2}$. If G is not cyclic then by Cauchy's theorem as p divides $|G| = p^2$, there is an element of G of order p. In fact for every $1 \neq x \in G$ since $|x|$ divides G and as $|x| \neq 1, p^2$ the only possibility is $|x| = p$. Take two distinct elements $1 \neq x$ and $1 \neq y$ of G and consider the subgroup of G generated by x and y, i.e. $\langle x, y \rangle$. Then as $|\langle x, y \rangle| > |x| = p$ it follows that $\langle x, y \rangle = G$. Also note that $\langle x \rangle \times \langle y \rangle \cong Z_p \times Z_p$. Clearly the map $\varphi : \langle x, y \rangle \to \langle x \rangle \times \langle y \rangle$ sending $x^a y^b \to (x^a, y^b)$ is an isomorphism. Hence in this case $G \cong Z_p \times Z_p$.

References:

[1] D.S. Dummit and R.M. Foote. *Abstract Algebra*, John Wiley (2007).

[2] Sureet Singh and Q. Zameeruddin, *Modern Algebra*, Vikas Pub. Eighth Ed. (2009).

[3] K. Tapp. *Matrix Groups for Undergraduates*, AMS (2005).