The Story of Magic Squares

Constructing Magic Squares

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Abstract

Magic squares are fascinating mathematical creations which were first known to people of China in early 2700 B.C and to Indian people in the twelfth century. Benjamin Franklin constructed three famous squares which have several interesting properties. Many people have tried to understand the method Franklin used to construct his squares (called Franklin squares) and many theories have been developed along these lines. In this project, we give an overview of the history and beliefs connected to the magic squares in early times and some of their methods of construction.

Contents

The Story of Magic Squares

1. Introduction.

A magic square of order *n* is an *n* x *n* square matrix (see ***A***, ***B***, ***C*** below) whose entries are nonnegative integers, such that the sum of the numbers in every row, in every column, and in each diagonal is the same number called the magic sum. The Lo Shu magic square is the oldest known magic square and its invention is attributed to Fuh Hi (2858-2738 B.C.), the mythical founder of the Chinese civilization. Although this may be the first record, it seems likely that others played with numbers to make the 'first' magic square [1]. Probably, many early humans discovered them independently. They may have played with piles of stones on a pattern in the sand or they may have stacked nuts on leaves laid out as a square grid. It seems somewhat improbable that it required a single mathematical genius in 2800 B.C. to develop the first simple 3x3 magic square.

4	9	2
3	5	7
8	1	6

A. *Lo Shu Square*

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

B. *Jaina Square*

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

C. *Durer Square*

Like chess and many of the problems, founded on the figure of the chess-board, the problem of constructing a magic square probably traces its origin to India. A 4×4 magic square is found in a Jaina inscription of the twelfth or thirteenth century in the city of Khajuraho, India. This square shows an advanced knowledge of magic squares because all the pandiagonals also add to the common magic sum. From India, the problem found its way among the Arabs, and by them it was brought to the Roman orient. It is recorded that as early as the ninth century magic squares were used by Arabian astrologers in their calculations of horoscopes etc. which might be the reason such squares are called *magic squares*. Their introduction into Europe appears to have been due to Moschopulus, who lived in Constantinople in the early part of the fifteenth century. The famous Cornelius Agrippa (1486-1535) constructed magic squares of the orders 3, 4, 5, 6, 7, 8, 9, which were associated by him with the seven astrological *planets*; namely, Saturn, Jupiter, Mars, the Sun, Venus, Mercury, and the Moon. A magic square engraved on a silver plate was sometimes prescribed as a charm against the plague and a magic square appears in a well known 1514 engraving by Albrecht Durer entitled *Melancholia* (see Fig. 1).

Figure 1: Magic square in a 1514 engraving by Albrecht Durer entitled Melancholia

It contains a Magic square. The two numbers in the middle of the fourth line represent the year in which he made the engraving.

Magic squares, in general, were considered as mystical objects with the power towards off evil and bring good fortune. Since then, certainly, many people in many nations have enjoyed, studied, and recorded magic squares. Magic squares are found in a number of cultures, including Egypt and India, engraved on stone or metal and worn as talismans, the belief being that magic squares had astrological and divinatory qualities, their usage ensuring longevity and prevention of diseases. For instance the *Kubera-Kolam* is a floor painting used in India which is in the form of a magic square of order three. It is essentially the same as the Lo Shu Square, but with 19 added to each number, giving a magic constant of 72. The 3x3 magic square was used as a part of rituals in India from vedic times, and continues to be used to this day. The Ganesh yantra is a 3x3 magic square.

In 1300, building on the work of the Arab Al-Buni, Greek Byzantine scholar Manuel Moschopoulos wrote a mathematical treatise on the subject of magic squares, leaving out the mysticism of his predecessors. Moschopoulos is thought to be the first Westerner to have written on the subject. In the 1450s the Italian Luca Pacioli studied magic squares and collected a large number of examples.

4	9	2
3	5	7
8	1	6

Saturn=15

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

Jupiter=34

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Mars=65 etc.

In about 1510 Heinrich Cornelius Agrippa wrote *De Occulta Philosophia*, drawing on the Hermetic and magical works of Marsilio Ficino and Pico della Mirandola, and in it he expounded on the magical virtues of seven magical squares of orders 3 to 9, each associated with one of the astrological planets. This book was very influential throughout Europe until the counter-reformation, and Agrippa's magic squares, sometimes called Kameas, continue to be used within modern ceremonial magic in much the same way as he first prescribed.

2. The Franklin Squares

Amongst our forebears, some very distinguished names have played with magic squares. Benjamin Franklin is one example. He played with the construction of magic squares in 1736-37 when he was a clerk of the Pennsylvania Assembly. The well-known squares F1 and F3, as well as the less familiar F2 that appear in Figure 2 below were constructed by Benjamin Franklin [2, 3].

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

F1

17	47	30	36	21	43	26	40
32	34	19	45	28	38	23	41
33	31	46	20	37	27	42	24
48	18	35	29	44	22	39	25
49	15	62	4	53	11	58	8
64	2	51	13	60	6	55	9
1	63	14	52	5	59	10	56
16	50	3	61	12	54	7	57

F2

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

F3

Figure 2: Franklin Squares F1, F2, and F3

In a letter to Peter Collinson he describes the properties of the 8×8 squares F1 as follows:

1. The entries of every row and column add to a common sum called the magic sum.

2. In every half-row and half-column the entries add to half the magic sum.

3. The entries of the main bent diagonals and all the bent diagonals parallel to it (see Fig. 3) add to the magic sum.

4. The four corner entries together with the four middle entries add to the magic sum.

Franklin mentions that the square F1 has five other curious properties but fails to list them. He also says, in the same letter, that the 16 × 16 square F3 has all the properties of the 8 × 8 square, but that in addition, every 4 × 4 sub square adds to the common magic sum. More is true about this square F3. Observe that every 2×2 sub square in F3 adds to one-fourth the magic sum. The 8 × 8 squares have magic sum 260 while the 16×16 square has magic sum 2056.

We define 8 × 8 *Franklin squares* to be squares with nonnegative integer entries that have the properties 1-4 above listed by Benjamin Franklin and the additional property that every 2×2 sub square adds to one-half the magic sum (see Figure 3).

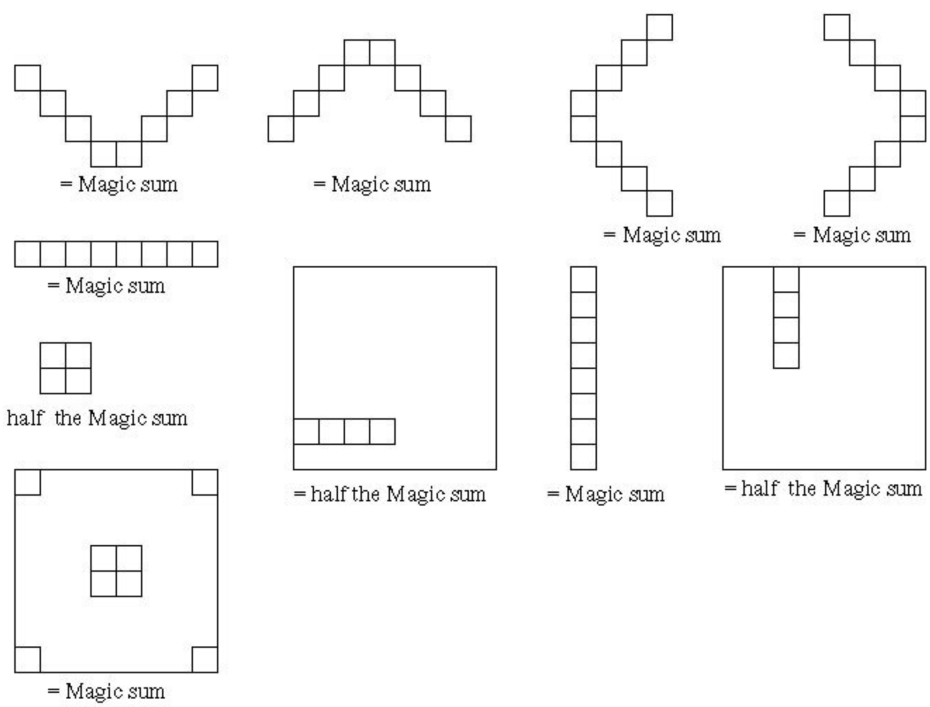

Figure 3: Defining properties of Franklin squares

The 8 × 8 squares constructed by Franklin have this extra property (this might be one of the stated curious properties to which Franklin was alluding in his letter). It is worth noticing that the fourth property listed by Benjamin Franklin becomes redundant with the assumption of this additional property.

Similarly, we define 16×16 *Franklin squares* to be 16×16 squares that have nonnegative integer entries with the property that all rows, columns, and bent diagonals add to the magic sum, the half-rows and half-columns add to one-half the magic sum, and the 2 × 2 sub squares add to one-fourth the magic sum. The 2 × 2 sub square property implies that every 4 × 4 sub square adds to the common magic sum.

3. Constructing Magic Squares

The mathematical theory of the construction of these squares was taken up in France only in the seventeenth century, and since then it has remained a favorite topic of study throughout the mathematical world. Constructing and enumerating magic squares are the two fundamental problems in the topic of magic squares.

Let $M_n(s)$ denote the number of n × n magic squares of magic sum *s*. In 1906, Mac Mahon enumerated magic squares of order 3 by the following formula:

$$M_3(s) = \begin{cases} \frac{2}{9}s^2 + \frac{2}{3}s + 1, & \text{if 3 divides } s \\ 0 & \text{otherwise} \end{cases}$$

We obtain all 3 × 3 and 4 × 4 magic squares and classify them using algebraic method of Ahmed [2]. To do so we first develop the necessary mathematical preliminaries as follows in the nex subsections.

3.1 Lattice points and Polyhedral Cones

In this section we briefly explain the algebra behind calculation of the magic squares that arises from the defining properties as mentioned in the previous section. We identify an *n × n* magic square by a square matrix of order *n* which can be further identified as a point in the vector space R^{n^2} to facilitate the normal algebraic operations on them. In this view, let $A = (x_{ij})_{n\times n}$, be a magic square of order *n*. Since sum of each row, column, and cross diagonals of *A* is same, using this defining property of *A* leads to the following set of linear equations in n^2 variables by subtracting the sum of the entries of the first row from all the rows and columns and the cross diagonals.

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{1j} = 0, \forall i = 2,...,n; \ \sum_{j=1}^{n} x_{ji} - \sum_{j=1}^{n} x_{1j} = 0, \forall i = 2,...,n,$$

$$\sum_{j=1}^{n} x_{jj} - \sum_{j=1}^{n} x_{j1} = 0; \ \sum_{j=1}^{n} x_{n-j+1,j} - \sum_{j=1}^{n} x_{j1} = 0,$$

which are $(n-1)+n+2 = 2n+1$ in number. This linear system can be represented as a homogeneous equation of the form

$$BX = 0$$

where $B = (B_{ij})_{(2n+1)\times n^2}$ and each entry of this matrix is an element of the set $\{-1,0,1\}$. Also, the matrix $X = (x_{ij})_{n^2\times 1}$.

3.11 Definition 1: *A subset C of* R^{n^2} *is said to be a cone if, for every pair of points u and v in C, the linear combination au+bv (for all nonnegative real numbers a and b) is also a point of C. A cone C is said to be pointed if origin is its only vertex. A cone C is said to be Polyhedral if there exists a matrix B s.t.* $C = \{Y \mid BY \le 0\}$. *A point of the cone C is said to be an integral point if its coordinates are integers.*

Observe that the sum of two magic squares is a magic square and an integral multiple of a magic square is also a magic square. Therefore the set of magic squares is the set of all integral points on the polyhedral cone $C_{M_n} = \{Y \mid BY = 0, Y_{ij} \ge 0\}$ which is a pointed cone.

For a given cone *C* let us call $S_C = C \cap Z^n$ of the integral points of *C* as the *semigroup of the cone C.*

3.12 Definition 2: *A Hilbert basis for a cone C is a finite set of points* $HB(C) \subset S_C = C \cap Z^n$ *such that each element of* S_C *is a linear combination of elements of* $HB(C)$ *with nonnegative integer coefficients.*

We quote the following result without proof for a polyhedral cone with rational points.

3.13 *Theorem 1:* *Each rational polyhedral cone C is generated by a Hilbert basis. If C is pointed, then there is a unique minimal integral Hilbert basis generating C.*

The minimal Hilbert basis of a pointed cone is unique and generates all the integral points on it. Once we know the minimal Hilbert basis *H* of the pointed cone consisting of magic squares, every magic square in *C* is the linear combination of elements of *H* with nonnegative integer coefficients. The elements of *H* are irreducible in a sense that each element of it cannot be expressed as a linear nonnegative integer combination of rest of the elements.

3.2 Minimal Hilbert bases for 3 ×3 and 4 ×4 Magic Squares

Ahmed [2] has carried out calculations to obtain the minimal Hilbert basis for the polyhedral cone of magic squares which for *n* = 3, is given by the following:

$$H_{M_3} = \left\{ \underset{H_1}{\begin{array}{|c|c|c|}\hline 1 & 0 & 2 \\ \hline 2 & 1 & 0 \\ \hline 0 & 2 & 1 \\ \hline\end{array}} \quad \underset{H_2}{\begin{array}{|c|c|c|}\hline 2 & 0 & 1 \\ \hline 0 & 1 & 2 \\ \hline 1 & 3 & 0 \\ \hline\end{array}} \quad \underset{H_3}{\begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline\end{array}} \quad \underset{H_4}{\begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline\end{array}} \quad \underset{H_5}{\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array}} \right\}$$

To generate the magic squares using the above minimal Hilbert basis, let us go through some examples:

$$H_3+H_4+3H_5=\begin{array}{|c|c|c|}\hline 4&9&2\\\hline 3&5&7\\\hline 8&1&6\\\hline\end{array},\ 3H_1+H_4+H_5=\begin{array}{|c|c|c|}\hline 4&3&8\\\hline 9&5&1\\\hline 2&7&6\\\hline\end{array}$$

$$3H_3+H_4+H_5=\begin{array}{|c|c|c|}\hline 2&9&4\\\hline 7&5&3\\\hline 6&1&8\\\hline\end{array},\ 3H_1+3H_3+2H_4+2H_5=\begin{array}{|c|c|c|}\hline 7&12&11\\\hline 14&10&6\\\hline 9&8&13\\\hline\end{array}$$

among which the first one is the Lo Shu magic square. We remark that different linear combinations may generate same magic square. For a given magic sum, the total number of such different magic squares is given by the Mac Mohan's formula as stated earlier.

The minimal Hilbert basis for 4 ×4 Magic Squares is given by the following

$$H_{M_4}=\begin{array}{|c|c|c|c|}\hline 0&0&1&0\\\hline 0&1&0&0\\\hline 0&0&0&1\\\hline 1&0&0&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&0&1&0\\\hline 1&0&0&0\\\hline 0&1&0&0\\\hline 0&0&0&1\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 0&0&1&0\\\hline 0&0&0&1\\\hline 0&1&0&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 1&0&0&0\\\hline 0&0&1&0\\\hline 0&1&0&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&1&0&0\\\hline 0&0&0&1\\\hline 0&0&1&0\\\hline 1&0&0&0\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 0&0&0&1\\\hline 0&1&0&0\\\hline 0&0&1&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 0&1&0&0\\\hline 1&0&0&0\\\hline 0&0&1&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&1&0&0\\\hline 0&0&1&0\\\hline 1&0&0&0\\\hline 0&0&0&1\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 1&0&0&1\\\hline 1&1&0&0\\\hline 0&1&0&1\\\hline 0&0&2&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&1&0&1\\\hline 1&1&0&0\\\hline 0&0&1&1\\\hline 1&0&1&0\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&0&1&0\\\hline 0&0&0&2\\\hline 0&1&1&0\\\hline 1&1&0&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&0&2&0\\\hline 0&1&0&1\\\hline 1&1&0&0\\\hline 1&0&0&1\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&0&1&1\\\hline 0&1&0&1\\\hline 1&0&1&0\\\hline 1&1&0&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 1&1&0&0\\\hline 0&1&1&0\\\hline 0&0&0&2\\\hline 1&0&1&0\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 1&0&0&1\\\hline 0&0&1&1\\\hline 1&0&1&0\\\hline 0&1&0&0\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 0&2&0&0\\\hline 1&0&1&0\\\hline 0&0&1&1\\\hline 1&0&0&1\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 1&1&0&0\\\hline 1&0&1&0\\\hline 0&1&0&1\\\hline 0&0&1&1\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&1&0&1\\\hline 2&0&0&0\\\hline 0&1&1&0\\\hline 0&0&1&1\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 1&0&1&0\\\hline 0&0&1&1\\\hline 1&1&0&0\\\hline 0&1&0&1\\\hline\end{array},\begin{array}{|c|c|c|c|}\hline 0&0&1&1\\\hline 0&1&1&0\\\hline 2&0&0&0\\\hline 0&1&0&1\\\hline\end{array},$$

$$=\{H_1,H_2,\ldots,H_{20}\}$$

Here, $H_1 + 4H_3 + 2H_4 + 8H_5 + 3H_6 + 12H_7 + 4H_8 =$

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

and

$2H_1 + 10H_3 + 5H_4 + 2H_5 + 6H_6 + 8H_7 + H_8 =$

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

,

which are the familiar historical Jaina and Durer magic squares respectively.

4. Three Older Methods of Constructing a Magic Square

We consider odd, even, and doubly even ordered magic squares which can be constructed for any bigger order directly by following a certain strategy of filling the numbers starting from 1. Let us describe these constructions in the following subsections.

4.1 Odd ordered Magic Squares

An odd order magic square is of the form $n = 2k + 1$. There are several methods of generating such magic squares for $k \geq 1$. Among them the staircase method (see Fig 4) is more direct and simpler. In this method the numbers are written in ascending numerical order as an upward diagonal to the right. When a filled square is reached the next number is placed vertically below its predecessor. This method was devised by De la Loubère when the 1 is placed in the middle column of the top row [4]. If the number 1 lies in the middle column on the row directly above the middle row it is known as the method of Bachet de Méziriac [5].

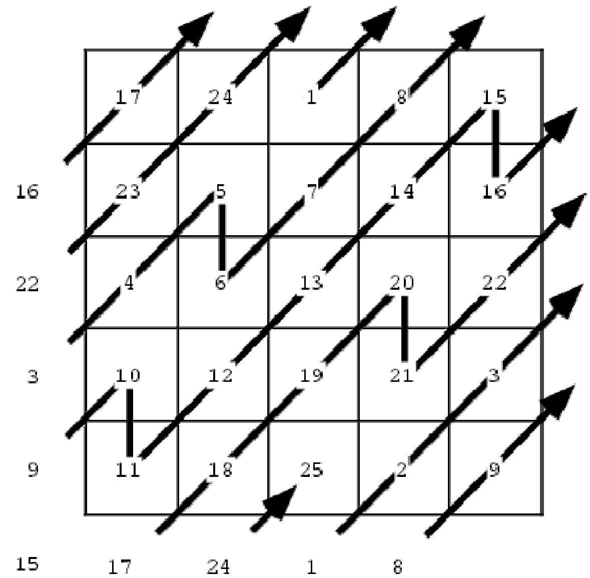

Figure 4: The staircase method for generating odd order magic squares

4.2 Doubly Even Magic Squares

A doubly even magic square is in the form $n = 4m$. One method of constructing this type of magic square, for $m \geq 1$, is the cross method (See Fig 5).

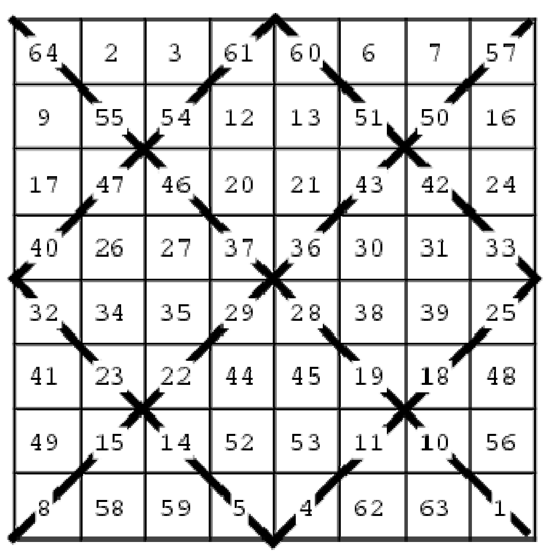

Figure 5: The cross method for generating doubly even magic squares

By writing all the numbers in order from the top left of a square to the bottom right, then drawing a cross through every 4x4 square, or sub-square of a larger square, and swapping the numbers along the diagonals of the cross, will yield a magic square.

4.3 Singly Even Magic Squares

A singly even magic square is of the form $4k+2,\ k \geq 1$. One method of construction is that of Ralph Strachey, to divide the square up into equal quarters. For example (see Fig. 6), in a 6x6 square, this will give four *3x3* squares. Each of these can then be formed using De la Loubère's method for odd order squares [4].

8	1	6	26	19	24
3	5	7	21	23	25
4	9	2	22	27	20
35	28	33	17	10	15
30	32	34	12	14	16
31	36	29	13	18	11

Figure 6: The De la Loubère's method for generating singly even magic square

We conclude with a remark that though there exist other different methods of obtaining the magic squares, never the less Ahmed [2, 3] has made an excellent attempt to construct and enumerate *all the magic squares* of order 4 and 8 using the Hilbert basis construction of the underlying polyhedral cones. We have not discussed the mathematical details of constructing the Hilbert basis with a hope that her work is easily readable and enjoyed by general audience who come across the magic of the magic squares.

5. References:

[1] Andrews, W. S., *Magic Squares and Cubes*, 2nd. ed., Dover, New York, 1960.

[2] Ahmed, M., *Ph.D. Thesis*, (2004).

[3] Ahmed, M., *How Many Squares Are There, Mr. Franklin?: Constructing and Enumerating Franklin Squares*., The American Mathematical Monthly, 111,5:394-410 (2004)

[4] Ball, W. W. R. *Mathematical recreations and essays*. London. Macmillan & Co Ltd. (1959)

[5] Kraitchik, M. *Mathematical Recreations*. London. George Allen & Unwin Ltd. (1960)