Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Notes on 'Locally convex Topological Vector Spaces'

In these notes we will introduce locally convex topological vector spaces and characterize them.

Recall that a seminorm p on a vector space $X(\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a map $p : X \to \mathbb{R}$ that satisfies nonnegativity, subadditivity, and homogeneity i.e. $p(x) \geq 0$, $p(x+y) \leq p(x) + p(y)$ and $p(tx) = |t|p(x)$ for all $x, y \in X$, $t \in \mathbb{K}$. Now let $\mathcal{P} = \{p_\alpha\}_{\alpha \in J}$ be an indexed family of seminorms on X.

Definition: An open ball centered at $x \in X$ and radius $r > 0$ with respect to a finite subcollection of seminorms $\mathcal{S} \subset \mathcal{P}$ is the set

$$V(x, \mathcal{S}, r) := \{y \in X \mid p(x-y) < r, \text{ for all } p \in \mathcal{S}\}.$$

Remark: Note that $V(x, \mathcal{S}, r) = x + V(0, \mathcal{S}, r)$. Also for any $x' \in V(x, \mathcal{S}, r)$, if we take $U(x', \mathcal{S}, \delta)$ such that $\delta := \min_{p \in \mathcal{S}}\{r - p(x - x')\}$ then for any $y \in U(x', \mathcal{S}, \delta)$, and $p \in \mathcal{S}$, we have

$$\begin{aligned} p(x-y) &\leq p(x-x') + p(x'-y) \\ &< p(x-x') + r - p(x-x') = r \end{aligned} \tag{0.1}$$

which shows that $y \in V(x, \mathcal{S}, r)$. We have proved that $U(x', \mathcal{S}, \delta) \subset V(x, \mathcal{S}, r)$. Thus $V(x, \mathcal{S}, r)$ contains an open ball around each of its points with respect to $\mathcal{S}$.

Remark: If $\mathcal{S}$ is a singleton $\{p\}$ where p is norm, then $V(x, p, r)$ is the open ball in X in the metric topology induced by p. We now topolozise X with the family of seminorms $\mathcal{P}$ as follows next.

Theorem 0.1. *Let $X(\mathbb{K})$ be a vector space and $\mathcal{P}$ be a family of seminorms on X. Let $\mathcal{N}_x$ be collection of all subsets of X of the form $V(x, \mathcal{S}, r)$, $r > 0$ for finite subset $\mathcal{S} \in \mathcal{P}$. Then the collection τ consisting of $\emptyset$ and all subsets of X of the form U such that for each point $y \in U$ there is a member $V \in \mathcal{N}_y$ such that $y \in V \subseteq U$ is a topology on X. This topology is compatible with the vector space structure of X. Moreover, each $p \in \mathcal{P}$ is continuous and that X is Hausdörff if and only if for each $0 \neq x \in X$ there is a $p \in \mathcal{P}$ satisfying $p(x) \neq 0$.*

Proof. Clearly $X \in \tau$. Let $\{U_\alpha\}_{\alpha \in J}$ be a family of open sets in X and set $U = \cup_{\alpha \in J} U_\alpha$. For any $x \in U$, there is a $\beta \in J$ such that $x \in U_\beta \in \tau$. So there is a $V \in \mathcal{N}_x$ such that $x \in V \subset U_\beta \subset U$. This proves that $U \in \tau$. Now let $A,\ B \in \tau$ and that $y \in A \cap B$. Choose $V(y, \mathcal{S}_1, r_1) \subset A$ and $V(y, \mathcal{S}_2, r_2) \subset B$ for some $r_1, r_2 > 0$ and finite subsets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{P}$. Then the open ball $V(y, \mathcal{S}_1 \cup \mathcal{S}_2, \delta) \subset A \cap B$ where $\delta := \min\{r_1, r_2\}$ and $V(y, \mathcal{S}_1 \cup \mathcal{S}_2, \delta) \in \mathcal{N}_y$. Hence $A \cap B \in \tau$. This verifies that τ is a topology on X.
For compatibility of τ, we need to show continuity of $+$ and $\cdot$. So consider for any $x \times y \in X \times X$, and open set $W \ni x + y$ in X. Then there is a $V(x+y, \mathcal{S}, r) \in \mathcal{N}_{x+y}$ such that $V(x+y, \mathcal{S}, r) \subset W$. Then the inverse image of $V(x+y, \mathcal{S}, r)$ under $+$ contains the open set $U := V(x, \mathcal{S}, r/2) \times V(x, \mathcal{S}, r/2) \ni x \times y$ since for any $z \times t \in U$ we have for all $p \in \mathcal{S}$, $p(x+y-(z+t)) \leq p(x-z) + p(y-t) < r/2 + r/2 = r$ which shows that $z + t \in W$. This proves continuity of $+$.
For establishing continuity of scalar multiplication, let $t \times x \in \mathbb{K} \times X$ and $W \ni tx$ be open. Then there is an open ball $V(tx, \mathcal{S}, r) \subset W$. Define $\delta := \min_{p \in \mathcal{S}}\{r - p(x)\}$. Then the open set $D(t, 1) \times V(x, \mathcal{S}, \delta) \ni t \times x$ where $D(t, 1) := \{s \in \mathbb{K} \mid |t - s| < 1\}$, such that $t'V(x, \mathcal{S}, \delta) \subset V(tx, \mathcal{S}, r)$ because for any $t' \times x' \in D(t, 1) \times V(x, \mathcal{S}, \delta)$ and all $p \in \mathcal{S}$ we have $p(tx - t'x') = p(tx - t'x + t'x - t'x') \leq |t - t'|p(x) + |t'|p(x - x') < p(x) + r - p(x) = r$ so that $x't' \in V(x, \mathcal{S}, r)$.
Continuity of p follows since for any $x \in X$ and basis element $I = (p(x) - \epsilon, p(x) + \epsilon)$ the inverse image $p^{-1}(I) \supset V(x, p, \epsilon) \ni x$. To see this for any $y \in V(x, p, \epsilon)$ we note that $|p(x) - p(y)| \leq p(x - y) < \epsilon$; this shows that $p(y) \in I$.
Finally, if X is Hausdörff and $x \neq 0$, then there exist disjoint open balls $V(x, \mathcal{S}_1, r)$ and $V(0, \mathcal{S}_2, s)$. Then as $x \notin V(0, \mathcal{S}_2, s)$ it follows that $p(x) \geq s > 0$ for all $p \in \mathcal{S}_2$. Conversely, let $x \neq y$ be two points in X. Then $x - y \neq 0$ and by hypothesis there is a $p \in \mathcal{P}$ such that $p(x-y) > 0$. Define $V(x, p, p(x-y))$ and $V(y, p, p(x-y))$ which are disjoint open sets containing x and y respectively. □

Remark: Each $\mathcal{N}_x$ in the preceding theorem forms a local base at x as is clear from the very first remark above.

Definition: The topology τ as in preceding theorem is called the vector space topology on X induced by family of seminorms $\mathcal{P}$.

Example: Let τ be vector space topology on X induced by a family of seminorms $\mathcal{P}$ on X. Let τ_1 be the vector space topology on $X \times X$ induced by the family of seminorms $\mathcal{P} \times \mathcal{P}$ defined by $(p \times q)(x \times y) := p(x) + q(y)$

[1]E-mail: sonumaths@gmail.com; Web page: https://sites.google.com/site/sonumaths2/

for all $p, q, \in \mathcal{P}$ and $x, y \in X$ and $V(x \times y, \mathcal{S}_1 \times \mathcal{S}_2, r) := \{z \times t \in X \times X \mid (p \times q)(x \times y) < r, \ \forall\, p \times q \in \mathcal{S}_1 \times \mathcal{S}_2\}$. Let τ_2 be product topology on $X \times X$. We will prove that $\tau_1 = \tau_2$.

So for any $x \times y \in X \times X$, let $V(x, \mathcal{S}_1, r) \times V(y, \mathcal{S}_2, s)$ be a basis element in τ_2 containing $x \times y$. Then the open ball $V(x \times y, \mathcal{S}_1 \times \mathcal{S}_2, \delta) \subset V(x, \mathcal{S}_1, r) \times V(y, \mathcal{S}_2, s)$ where $\delta := \min\{r, s\}$. This proves that $\tau_2 \subseteq \tau_1$. For reverse inclusion take any open ball $V(z \times t, \mathcal{S}_1 \times \mathcal{S}_2, R)$ in τ_1. Then $x \times y \in V(x, \mathcal{S}_1, R/2) \times V(y, \mathcal{S}_2, R/2) \subset V(x \times y, \mathcal{S}_1 \times \mathcal{S}_2, R)$ which establishes $\tau_1 \subset \tau_2$. Thus $\tau_1 = \tau_2$.

Theorem 0.2. *The vector space topology τ on $X(\mathbb{K})$ induced by a family of seminorms $\mathcal{P}$ is the smallest "compatible" topology on X in which each $p \in \mathcal{P}$ is continuous at 0.*

Proof. Let τ' be the smallest compatible topology on X in which each $p \in \mathcal{P}$ is continuous at 0 and $\tau' \subset \tau$. We will prove that $\tau \subset \tau'$. So let $\mathcal{S}$ be a finite subset of $\mathcal{P}$ and $r > 0$. Consider

$$V(0, \mathcal{S}, r) = \bigcap_{p \in \mathcal{S}} p^{-1}(-\infty, r)$$

which must be open in τ' by continuity of p at 0 in τ'. This proves that $V(0, \mathcal{S}, r) \in \tau'$. Since translation is a homeomorphism, it follows that the translates $V(x, \mathcal{S}, r) = x + V(0, \mathcal{S}, r) \in \tau'$. Thus $\tau \subset \tau$. □

Remark: The vector space topology on X induced by a family of seminorms $\mathcal{P}$ is not necessarily the smallest topology on X in which each $p \in \mathcal{P}$. Continuity of p requires for every open set $(p(x) - \epsilon, p(x) + \epsilon)$ existence of an open set $V \ni x$ such that $p(V) \subset (p(x) - \epsilon, p(x) + \epsilon)$. As $p(x) = p(-x)$ for all $x \in X$, it follows that such a V contains $-x$ as well and so it is symmetrical about the point $x = 0$. Now let τ_1 be collection of all open sets in the vector space topology on X which are symmetrical about $x = 0$. It is easy to see that τ_1 is topology on X which is smaller than vector space topology. However, in τ_1, for $x \neq 0$, the sequence $(-1)^n x \times (-1)^{n+1} x \to \pm x \times \pm x$ if $+$ could be continuous then by sequence lemma (note that X is first countable!) $0 = (-1)^n x + (-1)^{n+1} x \to \pm x + \pm x \in \{0, x, \pm 2x\}$ which gives zero sequence converging to four points $0, x, \pm 2x$ a contradiction since for any $p \in \mathcal{P}$, the open set containing $\pm x$, $V(x, p, p(x)/2) \cup V(-x, p, p(x)/2) \not\ni 0$.

Theorem 0.3. *Let ρ be a seminorm on a topological vector space X. Then ρ is continuous if and only if ρ is continuous at* 0.

Proof. Enough to prove that continuity of ρ at 0 implies continuity of ρ at all $x \in X$. So let ρ be continuous at 0 and $0 \neq x \in X$. Let $\epsilon > 0$ be given. Since for $0 \in \mathbb{K}$, $\rho(0x) = 0\rho(x) = 0$ this gives $\rho(0) = 0$. Any subbasis element of $\mathbb{R}$ containing $\rho(x)$ is of the form $(-\infty, \rho(x) + \epsilon)$ or $(\rho(x) - \epsilon, \infty)$. In either case $(-\infty, \epsilon)$ and $(-\epsilon, \infty)$ are two subbasis elements containing $\rho(0) = 0$. By continuity of ρ at 0 there exist open sets U and V containing 0 in X such that $\rho(U) \subset (-\infty, \epsilon)$ and $\rho(V) \subset (-\epsilon, \infty)$. Then $x + U$ and $x + V$ are the required open sets containing x such that $\rho(x + U) \subset \rho(x) + \rho(U) \subset (-\infty, \epsilon)$ and $\rho(x + V) \subset (-\epsilon, \infty)$. This proves continuity of ρ at x. □