**Department of Mathematics**
Guru Nanak Dev University, Amritsar-143005
Mathematics-532/506 Topology
(Hints to Solutions and Tentative Marking Scheme)
December 01, 2009

## Unit I

1. (i) Note that $p$ is a point of $A$ or a limit point of $A$ implies that $p$ is a point of $\bar{A}$ or a limit point of $\bar{A}$ ($\because\ \bar{A} \supset A$). In either case $p \in \bar{A}$ as it is closed. We have proved that $A \cup A' \subset \bar{A}$. For reverse implication, we prove that the set $A \cup A'$ is closed. W.l.o.g. assume that $x$ be a limit point of $A'$. Then every open set $U_x$ containing $x$ intersects $A'$ in a limit point say $q \neq x$ of $A$. But then $x$ becomes a limit point of $A$ and hence $x \in A'$. We have shown that set $A \cup A'$ contains all of its limit points and therefore it is closed. It follows that $A \cup A' \supset \bar{A}$ as by definition $\bar{A}$ is the smallest closed set containing $A$. This completes the proof. (**8 Marks**)

   (ii) From (i), the definitions of closed set and $\bar{A}$, we see that $A$ is closed iff $A = \bar{A} = A \cup A'$ iff $A' \subset A$. (**2 Marks**)

2. Let $X$ be a nonempty set. Kuratowski closure operator is a map$^{-}$: $\wp(X) \to \wp(X)$ satisfying the following 4 axioms for all $A, B \in \wp(X)$:
   (a) $\bar{\emptyset} = \emptyset$ (b) $A \subset \bar{A}$ (c) $\bar{\bar{A}} = A$ (d) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ (**2 Marks**)
   Now define a family $\tau := \{U \subset X \mid \overline{X \sim U} = X \sim U\}$ of open sets $U$ of $X$.(**2 Marks**)
   **Claim**: $\tau$ is a topology on $X$ induced by the operator $^{-}$. $X \in \tau$ as $\overline{X \sim X} = \bar{\emptyset} = \emptyset = X \sim X$(by axiom (a). Similarly, $\overline{X \sim \emptyset} = \bar{X} \supset X$(by axiom (b)); but as $X$ is the maximal element of $\wp$ (by Zorn's lemma!), $\bar{X} = X = X \sim \emptyset$. Hence $\emptyset \in \tau$. Secondly, Let $U$ $V$ be in $\tau$. Then $\overline{X \sim (U \cap V)} = \overline{(X \sim U) \cup (X \sim V)} = \overline{X \sim U} \cup \overline{X \sim V} = (X \sim U) \cup (X \sim V) = X \sim (U \cap V)$. This shows that $U \cap V \in \tau$. Finally, let $(U_\alpha)_{\alpha \in J}$ for some index set $J$, be a family of open sets in $\tau$. Consider $\overline{X \sim \cup U_\alpha} = \overline{\cap (X \sim U_\alpha)} \subset \cap \overline{(X \sim U_\alpha)} = \cap (X \sim U_\alpha) \subset \overline{\cap (X \sim U_\alpha)}$ ($\because \cap (X \sim U_\alpha) \subset X \sim U_\alpha$ for each $\alpha$; now use axiom (b) to obtain $\overline{\cap (X \sim U_\alpha)} \subset \overline{X \sim U_\alpha}$ for each $\alpha$). Hence $\overline{X \sim \cup U_\alpha} = \cap (X \sim U_\alpha) = X \sim \cup U_\alpha$. This shows that arbitrary union of members of $\tau$ is in $\tau$. With this we have established that $\tau$ is a topology on $X$ (**2×3=6 Marks**)

3. Given $Y = (\mathbb{R}, \mathcal{U})$. Let $q \in \mathbb{R}$ s.t. $q \notin \mathbb{Q}$ and define $U := (-\infty,\ q)$, $V := (q,\ \infty)$ as open and disjoint rays in $\mathbb{R}$. Note that $U \cap \bar{V} = \bar{U} \cap V = \emptyset$. Then the open sets of $\mathbb{Q} \cap U$ and $\mathbb{Q} \cap V$ of $\mathbb{Q}$ (as a subspace of $\mathbb{R}$) form a separation of $\mathbb{Q}$ which proves that $\mathbb{Q}$ is not connected. (**8 Marks**)

   Component of $p$ in $\mathbb{Q}$ is just the singleton set $\{p\}$ as is only the connected subspace of $\mathbb{Q}$ containing the point $p$. (**2 Marks**)

4. Given $Y \subset X$ and the inclusion map $i : Y \to X$; $i(a) = a$ is continuous. Then for any topology $\tau$ on $Y$ the inverse image of any open subset $U$ of $X$ under $i$ is open in Y. But by definition $i^{-1}(U) = Y \cap U$ which is already open in the subspace topology on $Y$. This means $\tau$ contains the subspace topology. The result follows now. (**10 Marks**)

## Unit II

5. $\Rightarrow$Suppose $f : X \to Y$ is continuous. then $f(\bar{A}) = f(A \cup A') = f(A) \cup f(A') \subset f(A) \cup f(A)' = \overline{f(A)}$. It remains to prove that $f(A') \subset f(A)'$. Let $f(p) \in f(A')$. As $f$ is continuous, for every open $V \in Y$ containing $f(p)$, $f^{-1}(V) \ni p$ is open in $X$ and it intersects $A$ in say $q \neq p$. Then $f(p) \neq f(q) \in (V \cap f(A))$ as $V$ is arbitrary neighborhood of $f(p)$, this implies that $f(p)$ is a limit point of $f(A)$ and therefore $f(p) \in f(A)'$. We have proved that $f(A') \subset f(A)'$. (**5 Marks**)

   $\Leftarrow$ Conversely, suppose that $f(\bar{A}) \subset \overline{f(A)}$. Let $B$ be a closed set in $Y$. Define $A = f^{-1}(B)$.
   **Claim**: $A$ is closed in $X$ and hence $f$ is continuous. Observe that $A = f^{-1}(B) \Rightarrow f(A) = f \circ f^{-1}(B) \subset B$. Also if $x \in \bar{A}$, then $f(x) \in f(\bar{A}) \subset \overline{f(A)} = B$. This means that $x \in A$. But

then $\bar{A} \subset A$which is possible only if $\bar{A} = A$. This proves the claim. (**5 Marks**)

6. Define a map $f : Y \times \{a\} \to Y$ by $f(y \times a) = y$. Clearly, $f$ is well defined ($\because y_1 \times a = y_2 \times a \Leftrightarrow y_1 = y_2$) surjective map and also it is injective.Therefore $f$ is a bijection. For any open set $U$ of $Y$, the inverse image $f^{-1}(U) = U \times \{a\}$ which is open subset of $Y \times \{a\}$ in the product topology. This proves that $f$ is continuous.
(**6 Marks**)

   By definition of $f$ for any open subset $U \times \{a\}$ of $Y \times \{a\}$, $f(U \times \{a\}) = U$ which is open in $Y$. This establishes the continuity of $f^{-1}$. (**4 Marks**)

7. Given $f : X \to \prod_\alpha Y_\alpha$; $\pi_\beta : \prod_\alpha Y_\alpha \to Y_\beta$; $\pi_\beta \circ f : X \to Y_\beta$. Observe that the projection map $\pi_\beta \prod_\alpha Y_\alpha \to Y_\beta$ is always continuous because for each $\beta$ and any open $V_\beta$ in $Y_\beta$ the inverse image $\pi_\beta^{-1}(V_\beta) = \prod_\alpha U_\alpha$ where $U_\alpha = Y_\alpha$ for all $\alpha \neq \beta$ and $U_\beta = V_\beta$, is open in the product topology on $\prod_\alpha Y_\alpha$. Suppose first $f$ is continuous and let $U_\beta$ be open subset of $Y_\beta$. By continuity of the projection map $\pi_\beta^{-1}(U_\beta)$ is open in $\prod_\alpha Y_\alpha$ and by continuity of $f$ the inverse image $f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta)$ is open in $X$. This proves that $\pi_\beta \circ f$ is continuous. (**8 Marks**)

   Conversely if $\pi_\beta \circ f$ is continuous then for any open $U$ in $Y_\beta$, $f^{-1}(\pi_\beta^{-1}(U))$ is open in $X$ which gives that $f$ is continuous. (**2 Marks**)

8. Let $X$ and $Y$ be connected. Then from (**Sol. Q. 6**) it follows that for any point $x \in X$, $\{x\} \times Y \cong Y$ and hence it is connected as well. Fix a point $a \in X$ and a point $b \in Y$. For any $x \in X$, define $T_x := (X \times \{b\}) \cup (\{x\} \times Y)$ Then $T_x \subset X \times Y$ and it is connected because a union of connected sets having a point in common is connected and the point $x \times b$ is common in $X \times \{b\}$ and $\{x\} \times Y$ which are connected. Then $\cup_{x \in X} T_x$ is being union of connected sets with a point $a \times b$ in common, is connected and since for every $x \times y \in X \times Y$, $x \times y \in T_x$ this means $X \times Y \subset \cup_{x \in X} T_x \subset X \times Y$. This is possible only if $\cup_{x \in X} T_x = X \times Y$. This proves that $X \times Y$ is connected. (**4 Marks**)

   It remains to prove that "A union of connected sets having a point common is connected." For this consider a family of connected sets $\{U_\alpha\}_{\alpha \in J}$ for some index set $J$, having a common point $p$. If possible suppose $(C, D)$ be a separation for $\cup_{\alpha \in J} U_\alpha$. Then either $p \in C$ or $p \in D$. W.l.o.g. suppose that $p \in C$. Now if $U_\alpha$ for some $\alpha$, intersects $D$; being connected, it must entirely lie in $D$ but $p \notin D$ and $p \in U_alpha$ a contradiction to our supposition. (**2 Marks**)

   Conversely let $X \times Y$ be connected then $X$ and $Y$ are connected as being continuous images of the connected space $X \times Y$ under the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, respectively. (**2 Marks**)

   ($\because$ if $(A, B)$ is a separation of $X$ then $(\pi_1^{-1}(A), \pi_1^{-1}(B))$ is a separation of $X \times Y$ a contradiction to connectedness of $X \times Y$. Similarly the other case for $Y$.) (**2 Marks**)

## Unit III

9. We first prove the following **lemma 1**: "every subspace of a metrizable space is metrizable." Let $(X, d)$ be a metrizable space and $Y$ be a subspace of $(X, d)$. Then any element of a basis for the subspace topology of $Y$ is $Y \cap B_d(x, \epsilon)$ for some open ball $B_d(x, \epsilon)$ centered at $x$ and of radius $\epsilon$ in the metric space $(X, d)$. Define a metric $d' : Y \times Y \to \mathbb{R}$ by $d'(x, y) = d(x, y)$ for all $x, y \in Y$. Then $d'$ is a restriction of the map $d|_Y$ and therefore is a metric on $Y$. Then any basis element of the metric topology on $Y$ is given by $B_{d'}(y, \epsilon) = Y \cap B_d(y, \epsilon)$. This proves that the basis for the subspace topology on $Y$ is same as that for the metric topology on $Y$ induced by $d'$ hence $Y$ is metrizable. (**2 Marks**)

   Now we prove **lemma 2**: "a metrizable space is normal". Let $(X, d)$ be a metric space and let

$A, B$ be two disjoint closed sets in $(X, d)$. For each $a \in A$ choose $\epsilon_a > 0$ such that $B_d(a, \epsilon_a) \cap B = \emptyset$ (by closedness of $B$ and that $a \notin B$) and define $U = \cup_{a \in A} B_d(a, \epsilon_a/2)$. Similarly for each $b \in B$ choose $\epsilon_b > 0$ such that $B_d(b, \epsilon_b) \cap A = \emptyset$ and define $V = \cup_{b \in B} B_d(b, \epsilon_b/2)$. It is clear that $A \subset U$ and $B \subset V$. Also $U$ and $V$ being unions of open balls, are open subsets of $X$. We claim that $U \cap V = \emptyset$. If possible let $z \in U \cap V$. then $d(z, a) < \epsilon_a/2$ and $d(z, a) < \epsilon_b/2$ using triangle inequality, $d(a, b) \leq d(a, z) + d(z, b) < (\epsilon_a + \epsilon_b)/2$. If $\epsilon_a \leq \epsilon_b$ then $d(a, b) < \epsilon_b$. This shows that $a \in B$ a contradiction. Similarly $\epsilon_a \geq \epsilon_b$ will lead to a contradiction. We have proved that a metrizable space is always normal. (**2 Marks**)

By **lemma 1** and **lemma 2**, it follows that every subspace of a metrizable space is normal. Therefore by definition, such a space is completely normal. (**2 Marks**)

The space $\mathbb{R}_K$ under $K-$topology is $T_2$ because any two distinct points have disjoint open intervals containing them. $\mathbb{R}_K$ is not $T_3$ because the set $K$ is closed in $\mathbb{R}_K$ and does not contain 0 but these two can not be contained in two disjoined open subsets of $\mathbb{R}_K$. Proof of this is also required (**see Munkres pp. 197 Example 1**) (**4 Marks**)

10. We prove **lemma** : "A compact subspace of a $T_2$ space is closed." Let $Y$ be a compact subspace of a $T_2$ space $X$. We prove $X \sim Y$ is open. Let $x_0 \in X \sim Y$. For each $y \in Y$ choose disjoint neighborhoods $U_y$ of $x_0$ and $V_y$ of $y$ in $X$. Then the collection $\{V_y \mid y \in Y\}$ is a covering of $Y$ by sets open in $X$. By compactness of $Y$, there is a finite sub collection covering $Y$, say $V = V_{y_1} \cup V_{y_2} \cup \cdots V_{y_m} \supset Y$ for some positive integer $m$. Then the corresponding set $U = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_m}$ is disjoint from $Y$ and contains the point $x_0$. Therefore $U \subset X \sim Y$. We have shown that $x_0$ is an interior point of $X \sim Y$. This establishes that $X \sim Y$ is open. Hence $Y$ is closed. (**5 Marks**)

Now let $A, B$ be two disjoint compact subsets of $X$. Then by the preceding **lemma**, these are closed. Now apply lemma 1 by taking $Y = A$ to construct $V$ and $U \ni x_0$ of $X \sim A$ s.t. $U \cap V = \emptyset$ and $x_0 \in U \subset X \sim A$, $A \subset V$. As $B \subset X \sim A$ which is open in $X$ as $A$ is closed, the sets $X \sim A$ and $V$ so obtained are the two disjoint open sets containing $B$ and $A$ respectively. (**5 Marks**)

11. We took this statement as a definition of completely normal spaces. May be you could prove or take as a definition, an equivalent statement " $Y$ is completely normal iff for every pair $A, B$ of separated sets in $X$, there exist disjoint open sets containing them." **Marks will be given accordingly**.

12. Urysohn's Lemma is a standard result and marks will be given according to the various steps of the proof.

### Unit IV

13. We have proved it as a **lemma in Q. 10**.

14. This is again a generalized version of the standard Cantor's Intersection Theorem:
A topological space $X$ is compact if and only if for every family $\mathcal{A}$ of open sets covering $X$ has a finite subcollection still covering $X$. *In other words $X$ is compact if and only if in any family of open sets no finite subcollection covers $X$, then the corresponding family of open sets is not a covering of $X$.* Consider a family $\mathcal{C}$ of closed subsets of $X$. Define $\mathcal{A} := \{X - C \mid C \in \mathcal{C}\}$.
(**3 Marks**)
It follows from De'Morgan's laws that
(i) $\mathcal{A}$ is an open covering of $X \Leftrightarrow \mathcal{C}$ is a covering of $X$ by closed sets with $\cap_{C \in \mathcal{C}} C = \emptyset$. Which means $\mathcal{A}$ *is not an open covering of* $X \Leftrightarrow \mathcal{C}$ *is not a covering of* $X$ *by closed sets with* $\cap_{C \in \mathcal{C}} C = \emptyset$ $\Leftrightarrow \cap_{C \in \mathcal{C}} C \neq \emptyset$ (ii) A finite subcollection of the members of $\mathcal{A}$ covers $X \Leftrightarrow$ intersection of corresponding finitely many members of $\mathcal{C}$ is empty. This is same as saying $A$ *finite subcollection of*

*the members of $\mathcal{A}$ does not cover $X \Leftrightarrow$ intersection of corresponding finitely many members of $\mathcal{C} \neq \emptyset \Leftrightarrow \mathcal{C}$ satisfies F.I.P.* (**5 Marks**)

The final result follows now by combining the results (appropriately) shown as in italics. (**2 Marks**)

15. This is a version of Hein-Borel's theorem. The usual proof borrowed from analysis is sufficient (using Cantor's intersection theorem for nested closed and bounded intervals of real numbers). However if you have followed the proof for order topology in general then it would be even nicer! (**5+5 Marks**)

16. Since the members of every open cover $\mathcal{A}$ of $X$ are composed of arbitrary unions of finite intersections of the sub-basis elements, every open covering of $X$ by the sub-basis open sets is contained in $\mathcal{A}$. Since the former covering has a finite subcover for $X$ the same subcover is the finite subcover for the covering $\mathcal{A}$ (Because arbitrary unions of finite intersections of finite number of sets are finite in number). This proves the compactness of $X$.

## Unit V

17. Let $X$ be countably compact and $A$ be an infinite subset of $X$. Let $\{U_n\}$ be a countable family of open sets covering $X$. If possible, suppose $A$ has no limit point then $A$ is closed and therefore $X \sim A$ is open. For each $a \in A$ chose an open set $U_{n_a}$ from the family $\{U_n\}$ such that $a \in U_{n_a}$ and $U_{n_a} \cap A = \{a\}$. Then the collection of open sets consisting of $U_{n_a}$ for each $a \in A$ along with the open set $X \sim A$ is a countable open covering of $X$, which has a finite sub collection covering $X$ by countable compactness of $X$. But then only finitely many $U_{n_a}$ cover $X$ and their union contain only the finitely many elements of the kind $a \in A$. This proves that $A$ is finite. We have proved that if $X$ is countably compact then a set $A$ in $X$ having no limit point is always finite. This is equivalent to the statement asked in the question.

18. This is standard result regarding Hausdorffness of one point compactification of a locally compact Hausdorff space $X$. Marks will be given on the basis of correct steps followed. (**5+5 Marks**)

19. Let every sequence in $Y$ has a cluster point. Take a countable family of closed sets with the finite intersection property and let $x_n$ be a point in the intersection of the first $n$ sets. Let $x$ be a cluster point of this sequence. Let $U$ be an open set containing $x$. As $x$ is a cluster point, $U$ contains infinitely many points of the sequence. Then $U$ intersects every closed set in the collection. This holds for every open set $U$ containing $x$. Thus $x$ is in the closure of every closed set in the collection. This means $x$ is in every closed set. The closed sets intersect, and $Y$ becomes countably compact.

20. Every countably compact space is limit point compact (**by Sol. Q.17**) enough to prove that limit point compactness implies compactness in metric spaces. To prove this one needs to prove the following: "for a metric space $X$, $X$ is limit point compact implies $X$ is sequentially compact which implies $X$ is compact." Proof is lengthy (**see Munkres pp. 179 Theorem 28.2** ) and marks will be given on the basis of correct steps.

**References:**


1. J. Munkres. *Topology*, Prentice Hall, 2005.
2. J. L. Kelley. *General Topology*, Springer, 1991.
3. K. Jänich. *Topology*, Springer, 1984.


The question paper was quite easy and nicely prepared. Except for the question no. **20** which seems to be considerably lengthy, all questions were either easy or standard results from the point set topology.


Jitender Singh
`Email: sonumaths@gmail.com`