Defining power sums of n and $\varphi(n)$ integers

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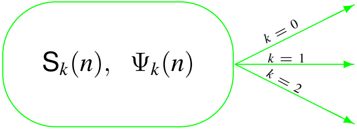
The Legacy of Srinivasa Ramanujan, DU

Power sums

Definition 1: Power sums for a positive integers n and $\varphi(n)$ are defined by

$$\mathsf{S}_k(n) = 1^k + 2^k + \cdots + n^k, \quad k \in \mathbb{C};$$

$$\Psi_k(n) = a_1^k + a_2^k + \cdots + a_{\varphi(n)}^k, \quad a_i \in \mathbb{Z}^+, \ a_i \leq n, \ \gcd(a, n) = 1.$$



$$n, \qquad \varphi(n)$$

$$\frac{n(n+1)}{2}, \qquad \frac{n\varphi(n)}{2}$$

$$\frac{n(n+1)(2n+1)}{6}, \quad \frac{\varphi(n)}{6}\left(2n^2 + \prod_{p|n} p\right)$$

Jakob Bernoulli and Euler

1. Jakob Bernoulli:

$$\mathbf{S}_k(n) = \frac{1}{k+1}\sum_{m=0}^{k}(-1)^m C(k+1,m)B_m n^{k+1-m},$$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n(n+1)} = 2, \sum_{n=1}^{\infty}\frac{n^2}{2^n} = 6, \sum_{n=1}^{\infty}\frac{n^3}{2^n} = 26$ etc.

The Basel Problem: Jakob Bernoulli failed to obtain in closed form:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

Later on Euler proved in (1735) that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{\pi^2}{6}.$$

$\Psi_k(n)$ from $S_k(n)$

Let $p,\ p_1,\ p_2, \cdots,\ p_r$ be distinct primes; $e,\ e_1,\ e_2,\ \cdots,\ e_r > 0$ and $k \geq 0$ be integers. Then

$$\Psi_k(p^e) = \mathbf{S}_k(p^e) - p^k\mathbf{S}_k(p^{e-1});$$

$$\Psi_k(n = p_1^{e_1}p_2^{e_2}) = \mathbf{S}_k(n) - p_1^k\mathbf{S}_k\left(\frac{n}{p_1}\right) - p_2^k\mathbf{S}_k\left(\frac{n}{p_2}\right) + p_1^kp_2^k\mathbf{S}_k\left(\frac{n}{p_1p_2}\right);$$

Theorem 1:

$$\Psi_k(n) = \mathbf{S}_k(n) - \sum p_i{}^k\mathbf{S}_k\left(\frac{n}{p_i}\right) + \cdots + (-1)^r p_1^kp_2^k\cdots p_r^k\mathbf{S}_k\left(\frac{n}{p_1p_2\cdots p_r}\right)$$

where $n = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$.

Corollaries

Cor. 1: $$\Psi_k(n) = \sum_{d|n} \mu(d) d^k \mathbf{S}_k\left(\frac{n}{d}\right); \quad \mathbf{S}_k(n) = n^k \sum_{d|n} \frac{\Psi_k(d)}{d^k}$$

where $\mu(d)$ is the Möbius function.

Cor. 2 $$\Psi_k(n) = \frac{n^{k+1}}{k+1} \sum_{m=0}^{[\frac{k}{2}]} C(k+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1})$$

where B_m is the $m-$th Bernoulli number defined via

$$\frac{y}{e^y - 1} = \sum_{m=0}^{\infty} B_m \frac{y^m}{m!};$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_{2m+1} = 0 \text{ for all } m = 1, 2, \cdots .etc.$$

$\Psi_k(n)$

$$\Psi_0(n) = \frac{n^{0+1}}{0+1}\sum_{m=0}^{[\frac{0}{2}]} C(0+1,2m)B_{2m}n^{-2m}\prod_{p|n}(1-p^{2m-1}) = \varphi(n);$$

$$\Psi_1(n) = \frac{n^{1+1}}{1+1}\sum_{m=0}^{[\frac{1}{2}]} C(1+1,2m)B_{2m}n^{-2m}\prod_{p|n}(1-p^{2m-1}) = n\varphi(n)/2;$$

$$\Psi_2(n) = \frac{n^{2+1}}{2+1}\sum_{m=0}^{[\frac{2}{2}]} C(2+1,2m)B_{2m}n^{-2m}\prod_{p|n}(1-p^{2m-1})$$

$$= n^2\varphi(n)/3 + B_2\, n\prod_{p|n}(1-p);$$

$\Psi_k(n)$

$$\Psi_3(n) = \frac{n^{3+1}}{3+1}\sum_{m=0}^{[\frac{3}{2}]} C(3+1,2m)B_{2m}n^{-2m}\prod_{p|n}(1-p^{2m-1})$$
$$= n^3\varphi(n)/4 + B_2\frac{3n^2}{2}\prod_{p|n}(1-p);$$
$$\Psi_4(n) = n^4\varphi(n)/5 + 2B_2n^3\prod_{p|n}(1-p) + B_4\, n\prod_{p|n}(1-p^3);$$
$$\Psi_5(n) = n^5\varphi(n)/6 + B_2\frac{5n^4}{2}\prod_{p|n}(1-p) + B_4\frac{5n^2}{2}\prod_{p|n}(1-p^3)$$ & so on.

The functional equations

Let x be a real variable and k be a nonnegative integer.

Definition 2: Define the power sum $S_k(x)$ *via* the generating function:

$$\frac{e^{(x+1)y} - e^y}{e^y - 1} := \sum_{k=0}^{\infty} \mathbf{S}_k(x)\frac{y^k}{k!}.$$

$$\mathbf{S}_k(x)\Big|_{\mathbb{Z}+} = \sum_{j=1}^{x} j^k;$$

$$\mathbf{S}_k(1) = 1 \;\; \forall \;\; k; \qquad \mathbf{S}_0(x) = x \;\; \forall \;\; x.$$

The functional equations

Let x be a real variable and k be a nonnegative integer.

Theorem 2:

1. $\mathbf{S}_k(x) = k\int_0^x \mathbf{S}_{k-1}(t)dt + xC_k,\ C_k = 1 - k\int_0^1 \mathbf{S}_{k-1}(t)dt$
2. $\mathbf{S}_k(x) = (-1)^{k+1}\mathbf{S}_k(-1-x),\ k \neq 0$

Idea:

1. $\frac{\partial}{\partial x}\left(\frac{e^{(x+1)y} - e^y}{e^y - 1}\right) = y\left(\frac{e^{(x+1)y} - e^y}{e^y - 1}\right) + \frac{ye^y}{e^y - 1}$
2. $\left(\frac{e^{(x+1)y} - e^y}{e^y - 1}\right) = -\left(\frac{e^{(1-1-x)(-y)} - e^{-y}}{e^{-y} - 1}\right) - 1.$

The functional equations

Let x be a real variable and k be a nonnegative integer.

Definition 3: Define using $\mathbf{S}_k(x)$, the power sum $\Psi_k(x,n)$ via

$$\Psi_k(x,n) = \sum_{d|n} \mu(d) d^k \mathbf{S}_k\left(\frac{x}{d}\right).$$

$$\sum_{d|n}\left[\mu(d)\frac{e^{(1+x/d)yd} - e^{yd}}{e^{yd}-1}\right] = \sum_{k=0}^{\infty} \Psi_k(x,n)\frac{y^k}{k!}; \quad \Psi_k(n,n) = \Psi_k(n).$$

Theorem 3:

$$\Psi_k(x,n) = k\int_0^x \Psi_{k-1}(x,n)dx + xC_k\prod_{p|n}(1-p^{k-1})$$

The zeroes of $\mathbf{S}_k(x)$

Definition 4: Call a representation of a real number r its simplest form if it contains at most one rational term. This rational term will be called as the rational part of r, and we shall denote it by $\mathbf{Q}(r)$.

Definition 5: Call $x = 0, -1$ as *trivial zeros* of $\mathbf{S}_k(x)$ for $k = 2, 3, 4, ...$ each with multiplicity at most 2.

$$r = 3 + \sqrt[3]{5},\ 1/2 + (5/2 + \sqrt[3]{5}),\ -2 + 1 + (4 + \sqrt[3]{5}); \mathbf{Q}(r) = 3$$

? *Let x be a nontrivial zero of $\mathbf{S}_k(x)$ for a fixed $k > 3$, then real(x) in its simplest form has rational part $\mathbf{Q}(x) = -\frac{1}{2}$.*

Alternating power sums

Definition 6: Call $\psi_k(p)$ the $k-$th Euler polynomial (Ramanujan [9])defined by

$$\frac{1}{e^y + p} = \sum_{k=0}^{\infty} \frac{(-1)^k \psi_k(p)}{(p+1)^{k+1}} \frac{y^k}{k!}, \; p \neq -1, \; |y| < |\log(p)|.$$

Theorem 4(Ramanujan): *For* $|p| < 1$ *and* $k \geq 0$,

$$(p+1)^{-1-k}\psi_k(p) = \sum_{n=0}^{\infty} (-p)^n (n+1)^k.$$

Alternating power sums

Motivated by Ramanujan's work we have the following:

Definition 7: Let $\alpha \neq 0, 1$. We say $\mathbf{E}_k(\alpha)$ is the $k-$th α-Euler number defined via the generating function

$$\frac{\alpha}{\alpha e^y - 1} := \sum_{k=0}^{\infty} \mathbf{E}_k(\alpha) \frac{y^k}{k!}, \ \alpha \neq 0, \ 1,$$

$$E_k(0) := 0.$$

Examples:

$$\mathbf{E}_0(\alpha) = \frac{\alpha}{\alpha - 1}; \ \mathbf{E}_1(\alpha) = -\frac{\alpha^2}{(1-\alpha)^2}$$

$$\mathbf{E}_2(\alpha) = \frac{\alpha^2(\alpha+1)}{(\alpha-1)^3}; \ \mathbf{E}_3(\alpha) = -\frac{\alpha^2(1+4\alpha+\alpha^2)}{(\alpha-1)^4} \ etc.$$

Alternating power sums

$$\frac{y}{e^y+1} = \frac{y}{e^y-1} - \frac{2y}{e^{2y}-1}$$

which gives

$$E_k(-1) = (1-2^k)\frac{B_{k+1}}{k+1};$$

$$\frac{\alpha}{\alpha e^y - 1} = \frac{1}{e^y + (-\alpha^{-1})}$$

$$\Rightarrow \quad (-1)^k \frac{\psi_k(-\alpha^{-1})}{(1-\alpha^{-1})^{k+1}} = \mathbf{E}_k(\alpha).$$

Theorem 5: *For* $\alpha \neq 0,\ 1,$

$$\mathbf{E}_{k+1}(\alpha) = \alpha^2 \frac{\partial}{\partial\alpha}(\mathbf{E}_k(\alpha)/\alpha)$$

$$\alpha^{-1}\mathbf{E}_k(\alpha) = (-1)^{k+1}\alpha\mathbf{E}_k(\alpha^{-1}),\ k \neq 0.$$

Alternating power sums

Definition 8: Call $\mathbf{S}_k(\alpha, x)$ as the $\alpha-$power sum defined by

$$\frac{\alpha^{x+1}e^{(x+1)y} - \alpha e^y}{\alpha e^y - 1} = \sum_{k=0}^{\infty} \mathbf{S}_k(\alpha, x)\frac{y^k}{k!}, \ \alpha \neq 1$$

1. $\mathbf{S}_0(\alpha, x) = \alpha\dfrac{1-\alpha^x}{1-\alpha}$, $\mathbf{S}_k(\alpha, 1) = \alpha$, $\lim_{\alpha\to 1^-} \mathbf{S}_k(\alpha, x) = \mathbf{S}_k(x)$.
2. When n is a positive integer
$$\mathbf{S}_k(\alpha, n) = \alpha + \alpha^2 2^k + \alpha^3 3^k + \cdots + \alpha^n n^k.$$
3. From definition
$$\mathbf{S}_k(\alpha, x) = \sum_{m=0}^{k} C(k, m)\mathbf{E}_m(\alpha)\{\alpha^x(1+x)^{k-m} - 1\}, \ \alpha \neq 1.$$

Alternating power sums

For $k > 0$, $\mathbf{S}_k(\alpha, -1) = 0$ and $\mathbf{S}_0(\alpha, -1) = -1$, this leads to the identity $(\alpha \neq 0, 1)$

$$\sum_{m=0}^{k} C(k,m)\mathbf{E}_m(\alpha) = \begin{cases} \alpha^{-1}\mathbf{E}_k(\alpha), & \text{for } k \neq 0 \\ \mathbf{E}_0(\alpha), & \text{for } k = 0. \end{cases} \tag{1}$$

Theorem 6:

$$\mathbf{S}_{k+1}(\alpha, x) = \alpha \frac{\partial}{\partial \alpha} \mathbf{S}_k(\alpha, x)$$

$$\frac{\alpha^{x+1} e^{(x+1)y} - \alpha e^y}{\alpha e^y - 1} = -\frac{(\alpha^{-1})^{1-1-x} e^{(1-1-x)(-y)} - \alpha^{-1} e^{-y}}{\alpha^{-1} e^{-y} - 1} - 1$$

$$\Rightarrow \mathbf{S}_k(\alpha, x) = (-1)^{k+1} \mathbf{S}_k(\alpha^{-1}, -1-x),\ k \neq 0$$

Abel-sum

As $\mathbf{S}_0(\alpha, x) = \alpha\dfrac{1-\alpha^x}{1-\alpha}$, and for $|\alpha| < 1$ $\alpha^x \to 0$ as $x \to \infty$, therefore, $\mathbf{S}_0(\alpha, x) \to \dfrac{\alpha}{1-\alpha} = -\mathbf{E}_0(\alpha)$ as $x \to \infty$. For $k > 0$ and $0 < |\alpha| < 1$,

$$\lim_{x\to\infty} \mathbf{S}_k(\alpha, x) = \lim_{x\to\infty} \sum_{m=0}^{k} C(k,m)\mathbf{E}_m(\alpha)\{\alpha^x(1+x)^{k-m} - 1\}$$

$$= -\sum_{m=0}^{k} C(k,m)\mathbf{E}_m(\alpha) = -\alpha^{-1}\mathbf{E}_k(\alpha).$$

The **Abel sum** for the divergent alternating series $\sum_{n=1}^{\infty}(-1)^n n^k$ is

$$\lim_{\alpha\to-1+} \sum_{n=1}^{\infty} \alpha^n n^k = \lim_{\alpha\to-1+} (-\alpha^{-1}\mathbf{E}_k(\alpha)) = \mathbf{E}_k(-1).$$

Abel-sum

s : a positive integer

ω_s : $(e^{\frac{2\pi\iota}{s}})$ $s-$ th root of unity

$$\frac{1}{(\alpha e^{y/s})^s - 1} = \frac{1}{s}\left\{\frac{1}{\alpha e^{y/s} - 1} + \frac{1}{\alpha\omega_s e^{y/s} - 1} + \cdots + \frac{1}{\alpha\omega_s^{s-1} e^{y/s} - 1}\right\}$$

$$\mathbf{E}_k(\alpha^s) = \frac{\alpha^{s-1}}{s^{k+1}}\{\mathbf{E}_k(\alpha) + \omega_s^{-1}\mathbf{E}_k(\alpha\omega_s) + \cdots + \omega_s^{-(s-1)}\mathbf{E}_k(\alpha\omega_s^{s-1})\}$$

$$\mathbf{E}_k(\alpha^2) = \frac{\alpha}{2^{k+1}}(\mathbf{E}_k(\alpha) - \mathbf{E}_k(-\alpha)).$$

Definition 9: $\mathbf{E}_k(1) := \frac{1}{1-2^{k+1}}\mathbf{E}_k(-1)$

$$\mathbf{E}_k(1) = \frac{1-2^k}{1-2^{k+1}}\frac{B_{k+1}}{k+1} = \frac{1-2^k}{1-2^{k+1}}\zeta(-k).$$

Extending definition of $\mathbf{S}_k(\alpha, x)$

Theorem 6 motivates the following:

Definition 10: $\mathbf{S}_k(\alpha, x) := \int_0^\alpha t^{-1}\mathbf{S}_{k+1}(t, x)dt, \ k = 0, \ \pm 1, \pm 2\ldots$

$$\lim_{x\to\infty} \mathbf{S}_{-1}(\alpha, x) = -\log|\alpha - 1|$$

$$\lim_{x\to\infty} \mathbf{S}_{-2}(\alpha, x) = -\log(\alpha)\log(1-\alpha) + \zeta(2) - \lim_{x\to\infty} \mathbf{S}_{-2}(1-\alpha, x),$$

$$\lim_{\alpha\to 1^-}\lim_{x\to\infty} \mathbf{S}_{-2}(\alpha, x) = \int_0^1 \frac{\log(t)}{1-t}dt = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \zeta(2)$$

Euler formula:

$$\lim_{x\to\infty} \mathbf{S}_{-2}(1/2, x) = -\frac{(\log(2))^2}{2} + \frac{1}{2}\zeta(2)$$

Extending definition of $\mathbf{S}_k(\alpha, x)$

$$\lim_{x\to\infty} \mathbf{S}_{-3}(\alpha, x) = \frac{1}{2}\int_0^{\alpha} \frac{(\log(t))^2}{1-t}dt + \log(\alpha)\lim_{x\to\infty} \mathbf{S}_{-2}(\alpha, x) - \frac{(\log(\alpha))^2}{2}\lim_{x\to\infty} \mathbf{S}_{-1}(\alpha, x) \tag{2}$$

Theorem 7: *For* $0 < \alpha < 1$ *and* $k = 1, 2, 3, \ldots$

$$\lim_{x\to\infty} \mathbf{S}_{-k}(\alpha, x) = \frac{(-1)^{k-1}}{(k-1)!}\int_0^{\alpha} \frac{(\log(t))^{k-1}}{1-t}dt - \sum_{\beta=1}^{k-1}(-1)^{\beta}\frac{(\log(\alpha))^{\beta}}{(\beta)!}\lim_{x\to\infty} \mathbf{S}_{-k+\beta}(\alpha, x)$$

Generalized power sums

Definition 11:

$$\frac{a^{1+x}e^{(b+cx)y} - ae^{by}}{ae^{cy} - 1} := \sum_{k=0}^{\infty} \mathbf{S}_k(a,b,c,x)\frac{y^k}{k!}, \; a,b,c \neq 0, \; a \neq 1 \quad (3)$$

For $x = 1, 2, \ldots$ we have

$$S_k(a,b,c,x) = ab^k + a^2(b+c)^k + a^3(b+2c)^k + \cdots + a^x(b+(x-1)c)^k.$$

$$S_k(a,1,1,x) = S_k(a,x); \; S_0(a,b,c,x) = a\frac{1-a^x}{1-a}$$

We are interested in the case when $a \in (-1,1)$ and $x \to \infty$. So, for $|a| < 1$, let us denote $\lim_{x\to\infty} \mathbf{S}_k(a,b,c,x)$ by $\mathbf{S}_k(a,b,c,\infty)$.

Generalized power sums

Definition 12: Define the generalized power sum $\mathbf{S}_{-k}(a, b, c, \infty)$ for $k = 0, \pm 1, \pm 2, \pm 3 \ldots$ and $c \neq 0,\ a \neq 0,\ 1,$ real$(bc^{-1}) > 0$ via the functional equations:

$$(1)\ \mathbf{S}_0(a, b, c, \infty) = \frac{a}{1-a},$$

$$(2)\ (-k+1)\mathbf{S}_{-k}(a, b, c, \infty) = \frac{\partial}{\partial b}\mathbf{S}_{-k+1}(a, b, c, \infty),$$

$$(3)\ \mathbf{S}_{-k}(a, b, c, \infty) = \frac{c^{-1}}{a^{(bc^{-1}-1)}}\int_0^a t^{(bc^{-1}-2)}\mathbf{S}_{-k+1}(t, b, c, \infty)dt.$$

Theorem 14

$$S_{-k}(1^-, b, c, \infty) := \frac{(-c)^{-k+1}}{\Gamma(k)}\int_0^{1^-}\frac{t^{bc^{-1}-1}(\log t)^{k-1}}{1-t}dt$$

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