Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Major Test
M.Sc.(Hons.) Mathematics, Semester-II
May 23, 2011

Subject: Calculus on Manifolds **Max. Marks 50**
Course: MTL453 **Time allowed: 03 Hours**

Note: This question paper consists of 10 questions each carrying 5 marks.

1. (a) Define boundary of a $n-$singular cube c in $A \subset \mathbb{R}^n$. $(2\frac{1}{2})$

   (b) Check whether the characteristic function of the set $\mathbb{Q}$ of rational numbers is integrable ? $(2\frac{1}{2})$

2. Prove the Heine-Borel theorem: the subspace $[0,1] \subset \mathbb{R}$ in the usual topology of $\mathbb{R}$ is compact. (5)

3. Let $g : A \to \mathbb{R}^n$ be injective and continuously differentiable function such that $\det g'(x) \neq 0$ for all $x \in A \subset \mathbb{R}^n$. If $f : g(A) \to \mathbb{R}$ is integrable and $\mathcal{O}$ is an admissible open cover of A such that for each $U \in \mathcal{O}$, $\displaystyle\int_{g(U)} f = \int_U (f \circ g)|\det g'|$ then prove that $\displaystyle\int_{g(A)} f = \int_A (f \circ g)|\det g'|$. (5)

4. Let $\{v_1, ..., v_n\}$ be a basis for a vector space $V(\mathbb{R})$ and let $\{\varphi_1, ..., \varphi_n\}$ be the dual basis defined via $\varphi_i(v_j) := \delta_i^j$. Prove that the set of all $k-$fold tensor products $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ $1 \leq i_1, ..., i_k \leq n$, is a basis for the tensor space $\mathcal{T}^k(V)$ which has dimension n^k. (5)

5. Prove that the set of all wedge products $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ $1 \leq i_1, ..., i_k \leq n$, of the elements of dual basis of an $n-$dimensional vector space $V(\mathbb{R})$ is a basis for the the subspace $\Lambda^k(V) \subset \mathcal{T}^k(V)$ consisting of alternating $k-$tensors and this subspace has dimension $\dfrac{n!}{k!(n-k)!}$. (5)

6. If ω is a $k-$form and η is an $\ell-$form and $d\omega$ denotes the differential of ω, then prove that

   $$\text{(i)} \quad d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \quad \text{(ii)} \quad d(d\omega) = 0.$$ (5)

7. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable and $f^* : \Lambda^k(\mathbb{R}^m_{f(p)}) \to \Lambda^k(\mathbb{R}^n_p)$ be a linear transformation between the vector spaces of alternating$-k$ tensors over the tangent spaces based at points $f(p) \in \mathbb{R}^m$ and $p \in \mathbb{R}^n$ respectively, s.t. $(f^*\omega)(p) := f^*(\omega(f(p)))$. Prove that

   $$f^*(dx^i) = \sum_{j=1}^{n} D_j f^i dx^j.$$

   Further if $n = m$, then prove that

   $$f^*(h\ dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n) = (h \circ f)(\det f')dx^1 \wedge \cdots \wedge dx^n$$

   where $h : f(\mathbb{R}^n) \to \mathbb{R}$. (5)

8. Prove the Poincaré lemma: If $A \subset \mathbb{R}^n$ is an open star-shaped with respect to zero then every closed form on A is exact. (5)

9. Establish the Stokes theorem $\displaystyle\int_c d\omega = \int_{\partial c} \omega$ where ω is a $(k-1)-$ form on an open set $A \subset \mathbb{R}^k$ and c is a $k-$chain in A. (5)

10. Let $f : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p,\ p \leq n$ be a differentiable map such that

    $$\det[\big(D_{n-p+j} f^i(x,y)\big)_{p \times p}] \neq 0,$$

    whenever $f(x,y) = 0$. Prove that the set $f^{-1}(\{0\})$ is an $(n-p)-$dimensional manifold in R^n. Hence show that the unit sphere $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$ is an n-dimensional manifold in $\mathbb{R}^{n+1}$. (5)