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DEFINING POWER SUMS OF n AND $\varphi(n)$ INTEGERS

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Let n be a positive integer and $\varphi(n)$ denotes the Euler phi function. It is well known that the power sum of n can be evaluated in closed form in terms of n. Also, the sum of all those $\varphi(n)$ positive integers that are coprime to n and not exceeding n, is expressible in terms of n and $\varphi(n)$. Although such results already exist in literature, but here we have presented some new analytical results in these connections. Some functional and integral relations are derived for the general power sums.

Keywords: Power sum; Euler phi function; generating function; α-Euler number; zeta function.

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1. Introduction

The study of summing the power sums of positive integers has been of wide interest since the work of Bernoulli and Euler. Bernoulli summed the finite series $1 + 2^k + 3^k + \cdots + n^k$ for a nonnegative integer k. He also obtained the sums of certain infinite series like $\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2$, $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$, $\sum_{n=1}^{\infty} \frac{n^3}{2^n} = 26$ etc. Eventually, he turned his attention to the series of the form $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ but failed to sum this series. Later in 1735, Euler finally succeeded to prove that this series sums to $\frac{\pi^2}{6}$, where so many others had failed. Euler worked a lot on infinite series (see [1–5]).

In this article, we have studied finite power sums obtained for n and $\varphi(n)$ integers by defining them via some continuous functions. Some functional equations have been derived for them. The related infinite series have been studied by defining them using the functional equations satisfied by the finite "continuous" power sums.

2. Power Sums of n and $\varphi(n)$ Integers

Let $\mathbf{S}_k(n)$ denotes the power sum for a positive integer n i.e. $\mathbf{S}_k(n) = \sum_{j=1}^{n} j^k$.

Definition 1. For any real or complex k and a positive integer n, we define the power sum $\mathbf{\Psi}_k(n)$ of $\varphi(n)$ integers as,

$$\mathbf{\Psi}_k(n) = a_1^k + a_2^k + \cdots + a_{\varphi(n)}^k, \tag{1}$$

where $a_1, a_2, \ldots, a_{\varphi(n)}$ are the positive integers each of which is less than or equal to n and relatively prime to n.

We see that $\mathbf{\Psi}_0(n) = \varphi(n)$; if p is a prime number then $\mathbf{\Psi}_k(p) = \mathbf{S}_k(p-1)$. Also, by taking $k = 1$ in the identity $\sum_{i=1}^{\varphi(n)} a_i^k = \sum_{i=1}^{\varphi(n)} (n - a_i)^k$ since $(a_i, n) = 1$ implies $(n - a_i, n) = 1$, it is easy to check that $\mathbf{\Psi}_1(n) = n\varphi(n)/2$. This method does not work for evaluating $\mathbf{\Psi}_2(n)$. One method to evaluate $\mathbf{\Psi}_k(n)$ in general has been given in Apostol [6].

Definition 2. Let $\mathbf{\Theta}_k(n)$ denotes the sum of kth power of the positive integers $b_1, b_2, \ldots, b_{n-\varphi(n)}$, each of which is less than or equal to n and none is relatively prime to n i.e.

$$\mathbf{\Theta}_k(n) = b_1^k + b_2^k + \cdots + b_{n-\varphi(n)}^k. \tag{2}$$

Observe that $\mathbf{\Psi}_k(n) = \mathbf{S}_k(n) - \mathbf{\Theta}_k(n)$. It is well known that when k is a nonnegative integer, the power sum

$$\mathbf{S}_k(n) = \frac{1}{k+1} \sum_{m=0}^{k} (-1)^m C(k+1, m) B_m n^{k+1-m}, \tag{3}$$

where $C(k+1, m)$ is the binomial coefficient and B_m is the mth bernoulli number defined via the generating function [7, 8]

$$\frac{y}{e^y - 1} = \sum_{m=0}^{\infty} B_m \frac{y^m}{m!}. \tag{4}$$

We now state and prove our results that follow, to evaluate $\mathbf{\Psi}_k(n)$, given the prime decomposition of n.

Proposition 1. *For a prime p and a positive integer e, $\mathbf{\Psi}_k(p^e) = \mathbf{S}_k(p^e) - p^k \mathbf{S}_k(p^{e-1})$.*

Proof. Any positive integer a which is less than or equal to p^e and $(a, p^e) > 1$, is one among the following: $p, 2p, 3p, \ldots, p^e$. Therefore, $\mathbf{\Theta}_k(p^e) = p^k + 2^k p^k + 3^k p^k + \cdots + p^{ke} = p^k(1 + 2^k + 3^k + \cdots + p^{k(e-1)}) = p^k \mathbf{S}_k(p^{e-1})$. From this the result follows. □

Proposition 2. *Let p_1 and p_2 be two distinct primes and e_1, e_2 be positive integers. Let $n = p_1^{e_1} p_2^{e_2}$, then, $\mathbf{\Psi}_k(n) = \mathbf{S}_k(n) - p_1^k \mathbf{S}_k(n/p_1) - p_2^k \mathbf{S}_k(n/p_2) + p_1^k p_2^k \mathbf{S}_k(n/p_1 p_2)$.*

Proof. Any positive integer $a \leq n$ satisfying $(a, n) > 1$, must be divisible by p_1 or p_2 or p_1p_2. Thus, the values of a form the list $p_1, 2p_1, \ldots, p_1^{e_1}p_2^{e_2},\ p_2, 2p_2, \ldots, p_1^{e_1}p_2^{e_2}$. Note that each of the positive integers a which is divisible by p_1p_2, occurs twice in the above list. Therefore, $\mathbf{\Theta}_k(n) = p_1^k(1+2^k+\cdots+p_1^{k(e_1-1)}p_2^{ke_2}) +\ p_2^k(1+2^k+\cdots+p_1^{ke_1}p_2^{k(e_2-1)}) - p_1^kp_2^k(1+2^k+3^k+\cdots+p_1^{k(e-1)}p_2^{k(e_2-1)}) = p_1^k\mathbf{S}_k(\frac{n}{p_1}) + p_2^k\mathbf{S}_k(\frac{n}{p_2}) - p_1^kp_2^k\mathbf{S}_k(\frac{n}{p_1p_2})$. Thus, $\mathbf{\Psi}_k(n) = \mathbf{S}_k(n) - \mathbf{\Theta}_k(n) = \mathbf{S}_k(n) - p_1^k\mathbf{S}_k(\frac{n}{p_1}) - p_2^k\mathbf{S}_k(\frac{n}{p_2}) + p_1^kp_2^k\mathbf{S}_k(\frac{n}{p_1p_2})$. □

Theorem 3. *For a positive integer $n > 1$,*

$$\mathbf{\Psi}_k(n) = \mathbf{S}_k(n) - \sum p_i^k\mathbf{S}_k\left(\frac{n}{p_i}\right) + \sum p_i^kp_j^k\mathbf{S}_k\left(\frac{n}{p_ip_j}\right) - \cdots + (-1)^r p_1^kp_2^k\cdots p_r^k\mathbf{S}_k\left(\frac{n}{p_1p_2\cdots p_r}\right), \tag{5}$$

where $n = p_1^{e_1}p_2^{e_2}p_3^{e_3}\cdots p_r^{e_r}$, p_i are distinct prime devisors of n and e_i are positive integers. Here, in a term such as $\sum p_i^kp_j^k\mathbf{S}_k(\frac{n}{p_ip_j})$, it is understood that we consider all possible products $p_i^kp_j^k\mathbf{S}_k(\frac{n}{p_ip_j})$ containing distinct prime factors of n taken two at a time.

Proof. Each positive integer $a \leq n$, s.t. $(a, n) > 1$, must be divisible by at least one of the integers among the integers in the set X of all products of i distinct prime devisors of n for $i = 1, 2, \ldots, r$. Then for fixed i, j each a is of the form: xm, for $x \in X$, and $m = 1, 2, \ldots, n/x$. We see that the sum of kth power of the devisors > 1 of n is

$$\begin{aligned}\mathbf{\Theta}_k(n) &= \sum p_i^k(1+2^k+3^k+\cdots+(n/p_i)^k)\\ &\quad - \sum p_i^kp_j^k(1+2^k+3^k+\cdots+(n/p_ip_j)^k) + \cdots\\ &\quad + \sum(-1)^r p_1^kp_2^k\cdots p_r^k(1+2^k+3^k+\cdots+(n/p_1p_2\cdots p_r)^k)\\ &= \sum p_i^k\mathbf{S}_k\left(\frac{n}{p_i}\right) - \sum p_i^kp_j^k\mathbf{S}_k\left(\frac{n}{p_ip_j}\right) + \cdots\\ &\quad + (-1)^{r-1}p_1^kp_2^k\cdots p_r^k\mathbf{S}_k\left(\frac{n}{p_1p_2\cdots p_r}\right),\end{aligned}$$

Subtracting $\mathbf{\Theta}_k(n)$ from $\mathbf{S}_k(n)$, gives the asserted result (5). □

From above theorem we readily see that

$$\mathbf{\Psi}_k(n) = \sum_{d|n}\mu(d)d^k\mathbf{S}_k\left(\frac{n}{d}\right), \quad \text{also } \mathbf{S}_k(n) = n^k\sum_{d|n}\frac{\mathbf{\Psi}_k(d)}{d^k}, \tag{6}$$

where $\mu(n)$ is the Möbius function. An immediate corollary to Theorem 3 reads

Corollary 4.

$$\mathbf{\Psi}_k(n) = \frac{n^{k+1}}{k+1} \sum_{m=0}^{[\frac{k}{2}]} C(k+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1}). \tag{7}$$

The important Corollary 4 explicitly evaluates $\mathbf{\Psi}_k(n)$. For example, we have

$$\mathbf{\Psi}_0(n) = \frac{n^{0+1}}{0+1} \sum_{m=0}^{[\frac{0}{2}]} C(0+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1}) = \varphi(n),$$

$$\mathbf{\Psi}_1(n) = \frac{n^{1+1}}{1+1} \sum_{m=0}^{[\frac{1}{2}]} C(1+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1}) = n\varphi(n)/2,$$

$$\mathbf{\Psi}_2(n) = \frac{n^{2+1}}{2+1} \sum_{m=0}^{[\frac{2}{2}]} C(2+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1})$$

$$= n^2\varphi(n)/3 + B_2\ n \prod_{p|n} (1 - p),$$

$$\mathbf{\Psi}_3(n) = \frac{n^{3+1}}{3+1} \sum_{m=0}^{[\frac{3}{2}]} C(3+1, 2m) B_{2m} n^{-2m} \prod_{p|n} (1 - p^{2m-1})$$

$$= n^3\varphi(n)/4 + B_2 \frac{3n^2}{2} \prod_{p|n} (1 - p),$$

$$\mathbf{\Psi}_4(n) = n^4\varphi(n)/5 + 2B_2 n^3 \prod_{p|n} (1 - p) + B_4\ n \prod_{p|n} (1 - p^3),$$

$$\mathbf{\Psi}_5(n) = n^5\varphi(n)/6 + B_2 \frac{5n^4}{2} \prod_{p|n} (1 - p) + B_4 \frac{5n^2}{2} \prod_{p|n} (1 - p^3) \quad \text{and so on.}$$

2.1. *The functional equations*

We have seen that $\mathbf{\Psi}_k(n)$ is expressible in terms of the power sum $\mathbf{S}_k(n)$ and the prime divisors of n and vice-versa. This suggests that both these functions should satisfy analogous identities. We first investigate some properties of the "*continuous*" power sum defined as follows.

Definition 3. Define $\mathbf{S}_k(x)$ as a function of a real variable x via the generating function:

$$\frac{e^{(x+1)y} - e^y}{e^y - 1} = \sum_{k=0}^{\infty} \mathbf{S}_k(x) \frac{y^k}{k!}. \tag{8}$$

From the definition, we observe that $\mathbf{S}_k(x)$ restricted to the set of positive integers, is same as the sum $\sum_{j=1}^{x} j^k$. Also note that $\mathbf{S}_k(1) = 1$ for all k and $\mathbf{S}_0(x) = x$ for all x.

Theorem 5.

(a) $\mathbf{S}_k(x) = k\int_0^x \mathbf{S}_{k-1}(t)dt + xC_k, \quad C_k = 1 - k\int_0^1 \mathbf{S}_{k-1}(t)dt.$

(b) $\mathbf{S}_k(x) = (-1)^{k+1}\mathbf{S}_k(-1-x), \quad k \neq 0.$

Proof. To prove (a) it is enough to prove that $\frac{\partial}{\partial x}\mathbf{S}_k(x) = k\mathbf{S}_{k-1}(x) + C_k$. Observe that $\frac{\partial}{\partial x}[\frac{e^{(x+1)y}-e^y}{e^y-1}] = y\frac{e^{(x+1)y}-e^y}{e^y-1} + \frac{ye^y}{e^y-1}$. Therefore,

$$\sum_{k=0}^{\infty} \frac{\partial}{\partial x}\mathbf{S}_k(x)\frac{y^k}{k!} = \sum_{k=0}^{\infty} \mathbf{S}_k(x)\frac{y^{k+1}}{k!} + \sum_{k=0}^{\infty} c_k\frac{y^k}{k!} = \sum_{k=1}^{\infty} \mathbf{S}_{k-1}(x)\frac{y^k}{(k-1)!} + \sum_{k=0}^{\infty} c_k\frac{y^k}{k!}$$

where $\sum_{k=0}^{\infty} c_k\frac{y^k}{k!} = \frac{ye^y}{e^y-1}$. Comparing the coefficients of y^k on both sides we get $\frac{\partial}{\partial x}\mathbf{S}_0(x) = c_0$ and $\frac{\partial}{\partial x}\mathbf{S}_k(x) = k\mathbf{S}_{k-1}(x) + c_k$, for $k = 1, 2, 3, \ldots$, which on integration in the interval $[0, x]$ gives the desired result with $c_k = C_k$ using the fact that that $\mathbf{S}_k(1) = 1$ for all k and $\mathbf{S}_0(x) = x$.

Proof of (b) follows as

$$\frac{e^{(x+1)y} - e^y}{e^y - 1} = -\frac{e^{(1-1-x)(-y)} - e^{-y}}{e^{-y} - 1} - 1. \qquad \square$$

Observe that the coefficients C_k are related to bernoulli numbers via $C_0 = B_0$, $C_1 = B_1 + 1$, and $C_k = B_k$ for all $k = 2, 3, \ldots$. In other words C_k is the kth Bernoulli polynomial $B_m(z)$ evaluated at $z = 1$. Bernoulli polynomials are defined via

$$\frac{ye^{zy}}{e^y - 1} = \sum_{m=0}^{\infty} B_m(z)\frac{y^m}{m!}.$$

It is clear that $B_m(0) = B_m$.

We remark here that Howard [9] has also studied the power sums of integers using generating functions, but the difference is that in the present definitions, we have defined the continuous power sums directly via generating function. Except for different notations, the Theorem 5(a) is same as has been given in Howard [9, p. 251], for $P_k(z)$, Eq. (5.11) for $a = d = 1$. Since

$$\sum_{k=0}^{\infty} \mathbf{S}_k(x)\frac{y^k}{k!} = \frac{e^{(x+1)y} - e^y}{e^y - 1} = \frac{1}{y}\frac{ye^{(x+1)y} - ye^y}{e^y - 1} = \frac{1}{y}\sum_{k=0}^{\infty}\{B_k(x+1) - B_k(1)\}\frac{y^k}{k!},$$

it is straightforward that

$$\mathbf{S}_k(x) = \frac{1}{k+1}\{B_{k+1}(x+1) - B_{k+1}(1)\}. \tag{9}$$

Equation (9) shows that $\mathbf{S}_k(x)$ is expressible in terms of Bernoulli polynomials. Thus, the various properties of the power sums can be obtained from the properties of the well-known Bernoulli polynomials.

Turning to the power sum of $\varphi(n)$ integers, define

$$\boldsymbol{\Psi}_k(x,n) = \sum_{d|n} \mu(d) d^k \mathbf{S}_k\left(\frac{x}{d}\right). \tag{10}$$

Note that

$$\sum_{d|n}\left[\mu(d)\frac{e^{(1+x/d)yd} - e^{yd}}{e^{yd}-1}\right] = \sum_{k=0}^{\infty} \boldsymbol{\Psi}_k(x,n)\frac{y^k}{k!} \quad \text{and} \quad \boldsymbol{\Psi}_k(n,n) = \boldsymbol{\Psi}_k(n).$$

We have thus, the following result:

Theorem 6.

$$\boldsymbol{\Psi}_k(x,n) = k\int_0^x \boldsymbol{\Psi}_{k-1}(x,n)dx + xC_k \prod_{p|n}(1-p^{k-1}),$$

Proof is similar to that of Theorem 5(a) and we omit it.

3. Zeros of $\mathbf{S}_k(x)$

In this section, we shall quote a very interesting and exciting result related to the zeros of the continuous power sum $\mathbf{S}_k(x)$. Before we proceed, we need a couple of definitions.

Definition 4. We say that a representation of a real number r is its simplest form if and only if it contains at most one rational term. This rational term will be called as the rational part of r, and we shall denote it by $\mathbf{Q}(r)$.

For example, $3+\sqrt[3]{5}$, $1/2+(5/2+\sqrt[3]{5})$, $-2+1+(4+\sqrt[3]{5})$, are different representations or forms of same real number containing 2, 3, 4 terms, respectively. Among these forms only the first one i.e. $3+\sqrt[3]{5}$ is the simplest form with a rational part 3.

It is clear from the closed form expression for $\mathbf{S}_k(x)$ that $x = 0, -1$ are the zeros of $\mathbf{S}_k(x)$ for $k = 2, 3, 4, \ldots$ each with multiplicity at most 2. We shall call $x = 0, -1$ as the trivial zeros of $\mathbf{S}_k(x)$. Also, it appears from the Theorem 5(b) that for a fixed $k > 0$, if x is a zero of $\mathbf{S}_k(x)$ then $-1-x$ is also its zero. Moreover, the distribution of the nontrivial zeros of $\mathbf{S}_k(x)$ about $\mathbf{Q}(r)$ is similar to the distribution of the nontrivial zeros of the Euler–Riemann Zeta function $\zeta(s)$ about real(s). Our investigations intuitively force us to conjecture the following result:

Let x be a nontrivial zero of $\mathbf{S}_k(x)$ for a fixed $k > 3$, then real(x) in its simplest form has rational part $\mathbf{Q}(x) = -\frac{1}{2}$.

To observe the truthfulness of this result, we calculate the zeros of $\mathbf{S}_k(x)$ for a few values of k. We have the following table:

k	Zeros of $\mathbf{S}_k(x)$
0	0
1	$0,\ -1$
2	$0,\ -1,\ -\frac{1}{2}$
3	$0,\ 0,\ -1,\ -1$
4	$0,\ -1,\ -\frac{1}{2},\ -\frac{1}{2} \pm \frac{\sqrt{21}}{6}$
5	$0,\ 0,\ -1,\ -1,\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$
6	$0,\ -1,\ -\frac{1}{2},\ -\frac{1}{2} \pm \frac{\sqrt{3(9-2\sqrt{3}i)}}{6},\ -\frac{1}{2} \pm \frac{\sqrt{3(9+2\sqrt{3}i)}}{6}$
7	$0,\ 0,\ -1,\ -1,\ -\frac{1}{2} \pm \frac{\sqrt{3(11-4\sqrt{2}i)}}{6},\ -\frac{1}{2} \pm \frac{\sqrt{3(11+4\sqrt{2}i)}}{6}$

Clearly the rational part of the real part of every nontrivial zero x in the table is $\mathbf{Q}(x) = -\frac{1}{2}$. It is obvious that the distribution of the zeros of $\mathbf{S}_k(x)$ is essentially related to the mysterious Bernoulli numbers.

We define the continuous power sum $\mathbf{S}_k(x)$ in more general setting in the next section and study some functional equations satisfied by them.

4. Alternating Power Sums

The study of alternating power sums goes back to the famous work of Euler. Ramanujan also worked on alternating sums. In Ramanujan's notation [9] let $\psi_k(p)$ be kth Euler polynomial defined by

$$\frac{1}{e^y + p} = \sum_{k=0}^{\infty} \frac{(-1)^k \psi_k(p)}{(p+1)^{k+1}} \frac{y^k}{k!}, \quad p \neq -1, \quad |y| < |\log(p)|. \tag{11}$$

Following result is due to Ramanujan, which relates the infinite sum $-\sum_{n=1}^{\infty}(-p)^k n^k$ with the Euler polynomial $\psi_k(p)$.

Theorem 7 (Ramanujan). *For $|p| < 1$ and $k \geq 0$,*

$$(p+1)^{-1-k}\psi_k(p) = \sum_{n=0}^{\infty} (-p)^n (n+1)^k. \tag{12}$$

Let $\alpha \neq 0$ be a complex number.

Definition 5. We say $\mathbf{E}_m(\alpha)$ is the mth α-Euler number defined via the generating function

$$\frac{\alpha}{\alpha e^y - 1} = \sum_{k=0}^{\infty} \mathbf{E}_k(\alpha) \frac{y^k}{k!} \quad \text{for } \alpha \neq 1.$$

We shall define $\mathbf{E}_k(1)$ via $\mathbf{E}_k(-1)$ later on in the next subsection. First few α-Euler numbers are:

$$\mathbf{E}_0(\alpha) = \frac{\alpha}{\alpha-1},$$

$$\mathbf{E}_1(\alpha) = -\frac{\alpha^2}{(1-\alpha)^2},$$

$$\mathbf{E}_2(\alpha) = \frac{\alpha^2(\alpha+1)}{(\alpha-1)^3},$$

$$\mathbf{E}_3(\alpha) = -\frac{\alpha^2(1+4\alpha+\alpha^2)}{(\alpha-1)^4}.$$

Also, $\mathbf{E}_k(-1) = (1-2^k)\frac{B_{k+1}}{k+1}$. Since $\frac{\alpha}{\alpha e^y-1} = \frac{1}{e^y+(-\alpha^{-1})}$, we have $(-1)^k\psi_k(-\alpha^{-1})/(1-\alpha^{-1})^{k+1} = \mathbf{E}_k(\alpha)$.

Theorem 8. *For* $\alpha \neq 0, 1$,

(a) $\mathbf{E}_{k+1}(\alpha) = \alpha^2\frac{\partial}{\partial\alpha}(\mathbf{E}_k(\alpha)/\alpha)$,

(b) $\alpha^{-1}\mathbf{E}_k(\alpha) = (-1)^{k+1}\alpha\mathbf{E}_k(\alpha^{-1}),\ k \neq 0.$ (13)

Proof. (a) $\sum_{k=0}^{\infty}\alpha^2\frac{\partial}{\partial\alpha}(\mathbf{E}_k(\alpha)/\alpha)\frac{y^k}{k!} = \alpha^2\frac{\partial}{\partial\alpha}(\frac{1}{\alpha e^y-1}) = \alpha\frac{\partial}{\partial y}(\frac{1}{\alpha e^y-1}) = \alpha\sum_{k=1}^{\infty} \frac{\mathbf{E}_k(\alpha)}{\alpha}\frac{y^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty}\mathbf{E}_{k+1}(\alpha)\frac{y^k}{k!}$. The result follows now.

(b) It follows from the following: $\alpha^{-1}\frac{\alpha}{\alpha e^y-1} = -\alpha\frac{\alpha^{-1}}{\alpha^{-1}e^{-y}-1} - 1.$ □

Definition 6. Call $\mathbf{S}_k(\alpha, x)$ as the α-power sum defined by

$$\frac{\alpha^{x+1}e^{(x+1)y} - \alpha e^y}{\alpha e^y - 1} = \sum_{k=0}^{\infty}\mathbf{S}_k(\alpha, x)\frac{y^k}{k!}, \quad \alpha \neq 1. \tag{14}$$

We make few observations from the definition: $\mathbf{S}_0(\alpha, x) = \alpha\frac{1-\alpha^x}{1-\alpha}$, $\mathbf{S}_k(\alpha, 1) = \alpha$, $\lim_{\alpha\to1^-}\mathbf{S}_k(\alpha, x) = \mathbf{S}_k(x)$. Observe that when n is a positive integer

$$\mathbf{S}_k(\alpha, n) = \alpha + \alpha^2 2^k + \alpha^3 3^k + \cdots + \alpha^n n^k. \tag{15}$$

A general expression for $\mathbf{S}_k(\alpha, x)$ can be obtained from definition and is given by

$$\mathbf{S}_k(\alpha, x) = \sum_{m=0}^{k} C(k, m)\mathbf{E}_m(\alpha)\{\alpha^x(1+x)^{k-m} - 1\}, \quad \alpha \neq 1. \tag{16}$$

Also for $k > 0$ $\mathbf{S}_k(\alpha, -1) = 0$ and $\mathbf{S}_0(\alpha, -1) = -1$, this leads to the identity

$$\sum_{m=0}^{k} C(k, m)\mathbf{E}_m(\alpha) = \begin{cases} \alpha^{-1}\mathbf{E}_k(\alpha), & \text{for } k \neq 0 \\ \mathbf{E}_0(\alpha), & \text{for } k = 0. \end{cases} \tag{17}$$

Theorem 9.

(a) $\mathbf{S}_{k+1}(\alpha, x) = \alpha \frac{\partial}{\partial \alpha} \mathbf{S}_k(\alpha, x)$,
(b) $\mathbf{S}_k(\alpha, x) = (-1)^{k+1} \mathbf{S}_k(\alpha^{-1}, -1-x), \quad k \neq 0$.

Proof. (a) We have $\sum_{k=0}^{\infty} \alpha \frac{\partial}{\partial \alpha} \mathbf{S}_k(\alpha, x) \frac{y^k}{k!} = \alpha \frac{\partial}{\partial \alpha} (\frac{\alpha^{1+x} e^{(1+x)y} - \alpha e^y}{\alpha e^y - 1}) = \frac{\partial}{\partial y} (\frac{\alpha^{1+x} e^{(1+x)y} - \alpha e^y}{\alpha e^y - 1}) = \sum_{k=1}^{\infty} \mathbf{S}_k(\alpha, x) \frac{y^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \mathbf{S}_{k+1}(\alpha, x) \frac{y^k}{k!}$.
(b) It follows from $\frac{\alpha^{x+1} e^{(x+1)y} - \alpha e^y}{\alpha e^y - 1} = -\frac{(\alpha^{-1})^{1-1-x} e^{(1-1-x)(-y)} - \alpha^{-1} e^{-y}}{\alpha^{-1} e^{-y} - 1} - 1$. □

4.1. *Abel sum*

We now investigate the limiting power sum $\lim_{x \to \infty} \mathbf{S}_k(\alpha, x)$ which converges for $|\alpha| < 1$. As $\mathbf{S}_0(\alpha, x) = \alpha \frac{1-\alpha^x}{1-\alpha}$, and for $|\alpha| < 1$ $\alpha^x \to 0$ as $x \to \infty$, therefore $\mathbf{S}_0(\alpha, x) \to \frac{\alpha}{1-\alpha} = -\mathbf{E}_0(\alpha)$ as $x \to \infty$. For $k > 0$ and $|\alpha| < 1$,

$$\begin{aligned} \lim_{x \to \infty} \mathbf{S}_k(\alpha, x) &= \lim_{x \to \infty} \sum_{m=0}^{k} C(k, m) \mathbf{E}_m(\alpha) \{\alpha^x (1+x)^{k-m} - 1\} \\ &= -\sum_{m=0}^{k} C(k, m) \mathbf{E}_m(\alpha) \qquad (18) \\ &= -\alpha^{-1} \mathbf{E}_k(\alpha), \end{aligned}$$

where the last step has been obtained using (17). The Abel sum for the alternating series $\sum_{n=1}^{\infty} (-1)^n n^k$ is now given by

$$\lim_{\alpha \to -1^+} \sum_{n=1}^{\infty} \alpha^n n^k = \lim_{\alpha \to -1^+} (-\alpha^{-1} \mathbf{E}_k(\alpha)) = \mathbf{E}_k(-1). \qquad (19)$$

Theorem 10. *Let s be a positive integer and ω_s denotes an sth root of unity. Then*

$$\mathbf{E}_k(\alpha^s) = \frac{\alpha^{s-1}}{s^{k+1}} \{\mathbf{E}_k(\alpha) + \omega_s^{-1} \mathbf{E}_k(\alpha \omega_s) + \cdots + \omega_s^{-(s-1)} \mathbf{E}_k(\alpha \omega_s^{s-1})\}. \qquad (20)$$

Proof. Since

$$\frac{1}{(\alpha e^{y/s})^s - 1} = \frac{1}{s} \left\{ \frac{1}{\alpha e^{y/s} - 1} + \frac{1}{\alpha \omega_s e^{y/s} - 1} + \cdots + \frac{1}{\alpha \omega_s^{s-1} e^{y/s} - 1} \right\},$$

which is valid, the result follows at once if we appeal to the definition of α-Euler numbers. □

Corollary 11.

$$\mathbf{E}_k(\alpha^2) = \frac{\alpha}{2^{k+1}} (\mathbf{E}_k(\alpha) - \mathbf{E}_k(-\alpha)). \qquad (21)$$

Definition 7.

$$\mathbf{E}_k(1) = \frac{1}{1 - 2^{k+1}} \mathbf{E}_k(-1). \qquad (22)$$

Note that for a positive integer $n \lim_{\alpha\to 1^-}\lim_{n\to\infty}\mathbf{S}_k(-\alpha,n) =$ Abel sum $-\eta(-k)$ for the divergent series $-1+2^k-3^k+\cdots+\infty$.

If one takes the Theorem 9(a) as a definition for $\mathbf{S}_k(\alpha,x)$ for all integers k and $\alpha\neq 0$ a real number, then we have

$$\mathbf{S}_k(\alpha,x)=\int_0^\alpha t^{-1}\mathbf{S}_{k+1}(t,x)dt, \tag{23}$$

with $\mathbf{S}_k(0,x)=0$. We see that $\mathbf{S}_{-1}(\alpha,x)=\int_0^\alpha t^{-1}\mathbf{S}_0(t,x)dt=\int_0^\alpha \frac{t^x-1}{t-1}dt = -\log|\alpha-1|+\int_0^\alpha \frac{t^x}{1-t}dt$, and for $|\alpha|<1, \lim_{x\to\infty}\int_0^\alpha \frac{t^x}{1-t}dt=\int_0^\alpha \lim_{x\to\infty}\frac{t^x}{1-t}dt\to 0$. Therefore, $\lim_{x\to\infty}\mathbf{S}_{-1}(\alpha,x)=-\log|\alpha-1|$. Similarly, we see that

$$\begin{aligned}\lim_{x\to\infty}\mathbf{S}_{-2}(\alpha,x)&=-\int_0^\alpha\frac{\log(1-t)}{t}dt\\&=-\log(\alpha)\log(1-\alpha)+\zeta(2)-\lim_{x\to\infty}\mathbf{S}_{-2}(1-\alpha,x),\end{aligned}$$

where $\lim_{\alpha\to 1^-}\lim_{x\to\infty}\mathbf{S}_{-2}(\alpha,x)=\int_0^1\frac{\log(t)}{1-t}dt=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\zeta(2)$. Thus,

$$\lim_{x\to\infty}\mathbf{S}_{-2}(\alpha,x)+\lim_{x\to\infty}\mathbf{S}_{-2}(1-\alpha,x)=-\log(\alpha)\log(1-\alpha)+\zeta(2). \tag{24}$$

In particular when $\alpha=1/2$

$$\lim_{x\to\infty}\mathbf{S}_{-2}(1/2,x)=-\frac{(\log(2))^2}{2}+\frac{1}{2}\zeta(2). \tag{25}$$

For $k=-3$,

$$\begin{aligned}\lim_{x\to\infty}\mathbf{S}_{-3}(\alpha,x)=\frac{1}{2}\int_0^\alpha\frac{(\log(t))^2}{1-t}dt+\log(\alpha)\lim_{x\to\infty}\mathbf{S}_{-2}(\alpha,x)\\-\frac{(\log(\alpha))^2}{2}\lim_{x\to\infty}\mathbf{S}_{-1}(\alpha,x).\end{aligned} \tag{26}$$

In general, we have the following result:

Theorem 12. *For $|\alpha|<1$ and $k=1,2,3,\ldots$*

$$\begin{aligned}\lim_{x\to\infty}\mathbf{S}_{-k}(\alpha,x)=\frac{(-1)^{k-1}}{(k-1)!}\int_0^\alpha\frac{(\log(t))^{k-1}}{1-t}dt\\-\sum_{\beta=1}^{k-1}(-1)^\beta\frac{(\log(\alpha))^\beta}{(\beta)!}\lim_{x\to\infty}\mathbf{S}_{-k+\beta}(\alpha,x).\end{aligned} \tag{27}$$

Theorem 13. *For $|\alpha|<1$ and $k=1,2,3,\ldots$*

(a) $\sum_{n=1}^\infty\frac{1}{(1-\alpha^n)}\frac{1}{n^k}=\lim_{\alpha\to 1^-}\lim_{x\to\infty}\mathbf{S}_{-k}(\alpha,x)+\sum_{n=1}^\infty\lim_{x\to\infty}\mathbf{S}_{-k}(\alpha^n,x)$,

(b) $\int_0^\alpha\lim_{x\to\infty}\frac{\mathbf{S}_{1-2k}(\alpha,x)}{1-\alpha}d\alpha=\frac{1}{2}\sum_{n=1}^{2k-1}(-1)^{n-1}\lim_{x\to\infty}\{\mathbf{S}_{-n}(\alpha,x)\mathbf{S}_{-2k+n}(\alpha,x)\}$. (28)

4.2. *A more general consideration*

Definition 8. For a nonnegative integer k define a generalized power sum $\mathbf{S}_k(a,b,c,x)$ by

$$\frac{a^{1+x}e^{(b+cx)y} - ae^{by}}{ae^{cy}-1} = \sum_{k=0}^{\infty} \mathbf{S}_k(a,b,c,x)\frac{y^k}{k!}, \qquad a,b,c \neq 0, \quad a \neq 1. \tag{29}$$

We immediately see that $\mathbf{S}_k(a,1,1,x) = \mathbf{S}_k(a,x)$. Also, $\mathbf{S}_0(a,b,c,x) = a\frac{1-a^x}{1-a}$. For $a = 1$, and $a = -1$, we obtain the generating functions for power sums and alternate power sums respectively, studied by Howard [9]. Here we study a more general class of power sums.

For any integer k, we define $\mathbf{S}_k(a,b,c,x)$, through the functional equations satisfied by them rather than deriving the functional equations. We are interested in the case when $|a| < 1$ and $x \to \infty$. So, for $|a| < 1$, let us denote $\lim_{x\to\infty} \mathbf{S}_k(a,b,c,x)$ by $\mathbf{S}_k(a,b,c,\infty)$.

Definition 9. Define $\mathbf{S}_{-k}(a,b,c,\infty)$ for $k = 0, \pm1, \pm2, \pm3, \ldots$ and $c \neq 0$, $a \neq 0, 1$, $\mathrm{real}(bc^{-1}) > 0$ via the functional equations:

(1) $\mathbf{S}_0(a,b,c,\infty) = \frac{a}{1-a}$,

(2) $(-k+1)\mathbf{S}_{-k}(a,b,c,\infty) = \frac{\partial}{\partial b}\mathbf{S}_{-k+1}(a,b,c,\infty)$,

(3) $\mathbf{S}_{-k}(a,b,c,\infty) = \frac{c^{-1}}{a^{(bc^{-1}-1)}}\int_0^a t^{(bc^{-1}-2)}\mathbf{S}_{-k+1}(t,b,c,\infty)dt$.

Then we have

$$\begin{aligned}\mathbf{S}_{-1}(a,b,c,\infty) &= \frac{c^{-1}}{a^{(bc^{-1}-1)}}\int_0^a t^{(bc^{-1}-2)}\mathbf{S}_0(t,b,c,\infty)dt \\ &= \frac{c^{-1}}{a^{(bc^{-1}-1)}}\int_0^a \frac{t^{(bc^{-1}-1)}}{1-t}dt.\end{aligned} \tag{30}$$

$$\begin{aligned}\mathbf{S}_{-2}(a,b,c,\infty) &= -\frac{\partial}{\partial b}\mathbf{S}_{-1}(a,b,c,\infty) \\ &= -\frac{c^{-2}}{a^{(bc^{-1}-1)}}\int_0^a \frac{t^{(bc^{-1}-1)}\log(t)}{1-t}dt + c^{-1}\log(a)\mathbf{S}_{-1}(a,b,c,\infty).\end{aligned} \tag{31}$$

$$\begin{aligned}\mathbf{S}_{-3}(a,b,c,\infty) &= \frac{1}{2}\frac{\partial^2}{\partial b^2}\mathbf{S}_{-1}(a,b,c,\infty) \\ &= \frac{c^{-3}}{2a^{(bc^{-1}-1)}}\int_0^a \frac{t^{(bc^{-1}-1)}(\log(t))^2}{1-t}dt + c^{-1}\log(a)\mathbf{S}_{-2}(a,b,c,\infty) \\ &\quad - c^{-2}\frac{\log(a)^2}{2}\mathbf{S}_{-1}(a,b,c,\infty).\end{aligned} \tag{32}$$

Thus, in general we have

Theorem 14. *For* $|a| < 1$, $c \neq 0$, $\text{real}(bc^{-1}) > 0$, *and* $k = 1, 2, 3, \ldots$

$$\begin{aligned}\mathbf{S}_{-k}(a,b,c,\infty) &= -\frac{(-1)^{k-1}}{(k-1)!}\frac{\partial^{k-1}}{\partial b^{k-1}}\mathbf{S}_{-1}(a,b,c,\infty)\\ &= \frac{(-c)^{-k+1}}{(k-1)!a^{(bc^{-1}-1)}}\int_0^a \frac{t^{(bc^{-1}-1)}(\log(t))^{k-1}}{1-t}dt\\ &\quad - \sum_{\beta=1}^{k-1}\frac{(-c)^{-\beta}(\log(a))^{\beta}}{\beta!}\mathbf{S}_{-k+\beta}(a,b,c,\infty).\end{aligned}$$

In particular,

$$\mathbf{S}_{-k}(1^-,b,c,\infty) = \frac{(-c)^{-k+1}}{(k-1)!}\int_0^{1^-}\frac{t^{(bc^{-1}-1)}(\log(t))^{k-1}}{1-t}dt, \tag{33}$$

where $\mathbf{S}_{-k}(1^-,b,c,\infty)$ *stands for* $\lim_{a\to 1^-}\mathbf{S}_{-k}(a,b,c,\infty)$.

Theorem 14 gives us an opportunity to define $\mathbf{S}_{-k}(1^-,b,c,\infty)$ for any complex k with positive real part, as follows

$$\mathbf{S}_{-k}(1^-,b,c,\infty) = \frac{(-c)^{-k+1}}{\Gamma(k)}\int_0^{1^-}\frac{t^{(bc^{-1}-1)}(\log(t))^{k-1}}{1-t}dt, \tag{34}$$

where $\Gamma(k)$ is the complex gamma function.

We have seen that each time we begin with defining a finite power sum, obtain the functional equations satisfied by them and the same functional equations hold for the power sums related to infinite series.

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