Department of Mathematics
Guru Nanak Dev University, Amritsar-143005
Major Test
M.Sc.(Hons.) Mathematics, Semester-II
May 18, 2011

Subject: Differential Geometry — **Max. Marks 50**
Course: MTL455 — **Time allowed: 03 Hours**

Note: This question paper consists of 10 questions each carrying 5 marks. The summation convention is presumed for the tensorial notations.

1. Obtain the envelope of one-parameter family of spheres of constant radius b and their centers on a circle of radius $a > b$. (5)

2. Obtain first fundamental form of the hemisphere with parametric representation
$$x = \frac{4a^2u}{4a^2 + u^2 + v^2},\; y = \frac{4a^2v}{4a^2 + u^2 + v^2},\; z = a\frac{4a^2 - u^2 - v^2}{4a^2 + u^2 + v^2}$$
where (u, v) are the curvilinear coordinates of any point on it and $a > 0$. (5)

3. Prove that the second fundamental form of surface of revolution $x = f(u)\cos v$, $y = f(u)\sin v$, $z = h(u)$ is given by $\Pi = \dfrac{f'h'' - f''h'}{\sqrt{f'^2 + h'^2}}(du)^2 + \dfrac{fh'}{\sqrt{f'^2 + h'^2}}(dv)^2$. (5)

4. Prove that the surface area of a parametric surface $\mathbf{r} = \mathbf{r}(u^1, u^2)$ over a rectangle Ω is given by
$$S_\Omega = \int_\Omega \sqrt{g}\,du^1du^2$$
and is an invariant under transformation $(u^1, u^2) \to (u^{1'}, u^{2'})$. (5)

5. Define Gaussian mapping and Gaussian curvature K for a regular smooth surface. Prove that $K = \dfrac{b}{g}$ where b and g have their usual meanings. (5)

6. Prove that the Gaussian curvature K of a regular smooth surface at any point (u^1, u^2) is:
$$K = \frac{1}{g}\left[\frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u^2 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^1 \partial u^1} - (\Gamma^r_{11}\Gamma^s_{22} - \Gamma^r_{12}\Gamma^s_{12})g_{rs}\right]$$
where Γ^k_{ij} are the Christoffel symbols of second kind. (5)

7. Prove that the geodesic curvature of a curve of class C_2 on a C_2 surface is given by
$$K_g = \sqrt{g}\begin{vmatrix} \dfrac{du^1}{ds} & \dfrac{du^2}{ds} \\ \dfrac{d^2u^1}{ds^2} + \Gamma^1_{ij}\dfrac{du^i}{ds}\dfrac{du^j}{ds} & \dfrac{d^2u^2}{ds^2} + \Gamma^2_{ij}\dfrac{du^i}{ds}\dfrac{du^j}{ds} \end{vmatrix}$$
where s is the natural parameter on the curve. (5)

8. Define geodesic of a surface. Prove that geodesic on a plane surface is a straight line. (5)

9. Define integral curvature and genus of a smooth surface. Calculate the integral curvature of (i) sphere (ii) torus. (5)

10. Prove that the integral curvature of a closed orientated surface of class C_3 satisfies
$$\frac{1}{2\pi}\int_\Omega K\,dS = \chi$$
where χ is the Euler-Poincaré characteristic of the surface. (5)