CHAPTER

ONE

INTRODUCTION

I see it, but I don't believe it.

G. Cantor

At the basis of the distance concept lies, for example, the concept of convergent point sequences and their defined limits, and one can, by choosing these ideas as those fundamental to point set theory, eliminate the notions of distance.

F. Hausdörff

The continuous development of science has resulted in an increased interweaving of different disciplines to unfold the hidden analogies. The point-set topology is a theory about the development of general concepts analogous to the keywords, such as "closeness", "continuity", and "convergence". Point-set topology enables us a spatial (physical) imagination of a number of abstract and non-intuitive problems, which makes topological methods more effective and relatively simple.

Like other disciplines, point-set topology evolved out of the revolutionary development in geometry during the late nineteenth century. By then, Mathematicians had realized that, there are some geometric-properties of certain spaces, which do not depend upon distance measurement. For example, removing one point from the unit-circle $S^1 := \{(x, y) \mid x^2 + y^2 = 1\}$ in $\mathbb{R}^2$ leaves it as one 'piece' while removing one point from $\mathbb{R}$ results into 'two disjoint pieces'(see Fig. 1.1). This geometrical distinction between $\mathbb{R}$ and S^1 does not involve any distance measurement. The distinction can be explained in terms of the underlying abstraction, which is based on the topological concept of 'connectedness' and 'continuity'.

Another example comes from a common property shared by the convex-Polyhedra (e.g. tetrahedron, cube etc.), which look very different from one another geometrically. Let V, E, and F denote the number of vertices, edges, and faces of a convex-

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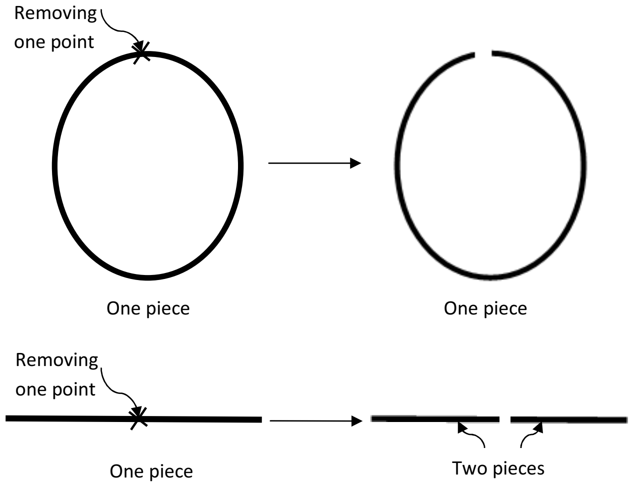


Figure 1.1

polyhedron. It can be readily seen that

$$V - E + F = 2. \tag{1.1}$$

The geometrical observation captured by (1.1) does not depend upon the size of the polyhedra. The quantity $\chi = V - E + F$ associated with a surface is called Euler-characteristic of the surface. A topological invariant of a surface is a geometrical property of the surface which is preserved under the operations of continuous stretching and bending, but not tearing the surface. The Euler-characteristic χ is a topological invariant of a surface, i.e., any two topologically same surfaces have the same Euler-characteristic. For example, the unit-sphere and the cube in $\mathbb{R}^3$ have $\chi = 2$, while $\chi = 0$ for the torus in $\mathbb{R}^3$. More generally, $\chi = 2 - g$, where g is the genus of the surface, i.e., the number of holes in the surface (e.g. the sphere has no hole, the torus has one hole). Study of the topological invariants such as the Euler-characteristic is central in topology.
Mathematical abstraction of point-set topology occurred in the beginning of twentieth-century with the work of Riesz (1909) who proposed first axiomatic definition of topology. The definition was based on a set of axioms of limit points with no concept of distance. Few years later, Felix Hausdörff (1914) defined topology via a set of axioms in terms of neighborhoods without any consideration of distance. The work of Riesz and Hausdörff together led to the modern definition of an abstract topological-space. In this chapter, we will learn some basic results on set-theory for later use.

1.1 Language of sets

The basic operations on sets such as union, intersection, difference, compliment, cartesian product will be presumed. So, for any two sets A and B; $A \cup B$ and $A \cap B$ denote their union and intersection, respectively. The difference of A and B is defined as the set $A - B = \{a \in A \mid a \notin B\}$, where the symbol '$\in$' means 'belongs to', and the symbol '$\notin$' means 'does not belong to'. The set $A \times B$ is the cartesian product of A and B. $A \subseteq B$ (or $B \supseteq A$) means A is a subset of B (or B contains A),i.e., each element of A is also an element of B. Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$. The notation $A \subsetneq B$ (or $B \supsetneq A$) means A is a proper subset of B. The set of all subsets of a given set X is called the power set of X denoted $\mathcal{P}(X)$. Empty set denoted $\emptyset$ is the set that does not contain any element. Thus, $\emptyset$ is a subset of every set.

Definition. Power set of a set X is the set of all subsets of X denoted $\mathcal{P}(X)$.

Definition. A function or map from a set A to a set B is a rule f that assigns to every element x of A, a unique-element $f(x)$ of B. We express this association by $f : A \to B$, which means f is a function from A to B. The set A is called the domain of f and the set of all values of f is called the range of f.

Definition. Let A and B be two sets and $f : A \to B$. The image of any subset $E \subseteq A$ under f is defined as the set $f(E) = \{f(x) \mid x \in E\}$. The inverse image of $F \subseteq B$ under f is defined as the set $f^{-1}(F) = \{x \in A \mid f(x) \in F\}$.

By *uniqueness* in the definition of function f from A to B above, we mean for each $x \in A$, the set $f(\{x\})$ is singleton. The function f is said to be well-defined if for $x = y$ implies $f(x) = f(y)$. The function f is said to be injective or one-one function if for every pair of points $x, y \in A$, $f(x) = f(y)$ implies $x = y$. The map f is said to be surjective or onto if for every $y \in B$, $f^{-1}(\{y\}) \neq \emptyset$. A function is said to be bijective if it is both injective as well as surjective.

Example. Let $\mathbb{Q}$ denote the set of all rational numbers. The map $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(p/q) = pq$ is not well defined since $1/2 = 2/4$ but $f(1/2) = 2 \neq 8 = f(2/4)$.

Definition. Two maps $f : A \to B$ and $g : A \to B$ are said to be equal, i.e., $f = g$ if $f(x) = g(x)$ for all $x \in A$.

Definition. The composition of two maps $f : A \to B$ and $g : B \to C$ is the map $g \circ f : A \to C$, such that $g \circ f(x) = g(f(x))$ for all $x \in A$.

Example. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 4, 9, 10\}$, and $C = \{0, 1, 3, 8, -2\}$ and $f : A \to B$, $g : B \to C$ are two functions such that $f(x) = x^2$ and $g(x) = x - 1$. Then the composition $(g \circ f) : A \to C$ such that $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 1$.

Definition. The identity map of a nonempty set A is defined as $Id_A : A \to A$ such that $Id_A(x) = x$ for all $x \in A$.

Definition. A function $f : A \to B$ is said to be invertible if there is a map $g : B \to A$ called inverse of f denoted f^{-1} such that $gof = Id_A$ and $f \circ g = Id_B$.

Example. The identity map $Id_A : A \to A,\ A \neq \emptyset$ is always invertible since it satisfies $Id_A \circ Id_A = Id_A$ so that $Id_A^{-1} = Id_A$.

Example. Let $\mathbb{C}$ denote the set of all complex numbers. The map $f : \mathbb{C} \to \mathbb{C}$, such that $f(z) = az + b$ is invertible if $a \neq 0$, with the inverse map define by $f^{-1}(w) = \dfrac{w-b}{a}$ since for all $w \in \mathbb{C}$

$$(f \circ f^{-1})(w) = f(f^{-1}(w)) = f\Big(\frac{w-b}{a}\Big) = a\frac{w-b}{a} + b = w. \tag{1.2}$$

Similarly, $(f^{-1} \circ f)(z) = z$ for all $z \in \mathbb{C}$.

Theorem 1. *A map $f : A \to B$ is invertible if and only if f is bijective.*

Proof. Let $f : A \to B$ is invertible and $g : B \to A$ is the inverse of f. To prove that f is bijective, we have to show that f is injective as well as surjective. As $g \circ f$ is well defined, $f(x) = f(y)$ implies $g \circ f(x) = g \circ f(y)$, which gives $Id_A(x) = Id_A(y)$ or $x = y$. Thus, f is injective. To prove that f is surjective, let $z \in B$. Then $g(z) \in A$ such that $f(g(z)) = z$ and f is surjective.
Conversely, let f is bijective. Then each element $y \in B$ is associated with exactly one element $x \in A$ such that $y = f(x)$ so that each $y \in B$ can be identified with $x \in A$. Define $g : B \to A$ by $g(f(x)) = x$ for all $f(x) \in B$ or $x \in A$. Then g is well-defined and, the map g is the inverse of f, since $g \circ f = Id_A$ by definition of g and $(f \circ g)(f(x)) = Id_B(f(x))$, which by equality of maps, gives $f \circ g = Id_B$. Thus, the map f is invertible. ■

Theorem 2. *Let X and Y are two sets and $f : X \to Y$. For $A,\ B \subseteq X$ and $P,\ Q \subseteq Y$, the following are true:*
(a) If $A \subseteq B$ then $f(A) \subseteq f(B)$; if $P \subseteq Q$ then $f^{-1}(P) \subseteq f^{-1}(Q)$
(b) $f^{-1}(f(A)) \supseteq A$ and $f(f^{-1}(P)) \subseteq P$
(c) $f(A \cup B) = f(A) \cup f(B)$
(d) $f(A \cap B) \subseteq f(A) \cap f(B)$, where equality holds when f is injective.
(e) $f^{-1}(P \cup Q) = f^{-1}(P) \cup f^{-1}(Q)$; $f^{-1}(P \cap Q) = f^{-1}(P) \cap f^{-1}(Q)$
(f) $f(X - A) \supseteq f(X) - f(A)$, where equality holds if f is injective.
(g) $f^{-1}(Y - P) = X - f^{-1}(P)$.

Proof. (a) obvious.
(b) For each $x \in A$, $f(x) \in f(A)$, i.e., $x \in f^{-1}(f(A))$, which proves that $A \subseteq f^{-1}(f(A))$.
For the second part, let $z \in f(f^{-1}(P))$; then $z = f(x)$ for some $x \in f^{-1}(P)$, which gives $z = f(x) \in P$. So, $f^{-1}(f(P)) \subseteq P$.
(c) Observe that $f(x) \in f(A \cup B)$ if and only if $x \in A \cup B$ if and only if either $x \in A$ or $x \in B$, i.e., either $f(x) \in f(A)$ or $f(x) \in f(B)$ if and only if $f(x) \in f(A) \cup f(B)$.

(d) Follows from the first part of (a), and if in addition, f is injective then for any $y \in (f(A) \cap f(B))$, there exist $a \in A$ and $b \in B$, such that $f(a) = y = f(b)$, which gives $a = b$ since f is injective. Thus, $a \in A \cap B$ or $y = f(a) \in f(A \cap B)$ and $(f(A) \cap f(B)) \subseteq f(A \cap B)$.
(e) Observe that $x \in f^{-1}(P \cup Q)$ if and only if $f(x) \in P \cup Q$ if and only if either $f(x) \in P$ or $f(x) \in Q$, i.e., either $x \in f^{-1}(P)$ or $x \in f^{-1}(Q)$ if and only if $x \in f^{-1}(P) \cup f^{-1}(Q)$. The other part can be proved the same way.
(f) Let $f(x) \in f(X) - f(A)$. Then $f(x) \notin f(A)$ or $x \notin A$. So $x \in (X - A)$ or $f(x) \in f(X - A)$. Thus $f(X) - f(A) \subseteq f(X - A)$. Further, if f is injective and $f(z) \in f(X - A)$, then $z \in (X - A)$ or $z \notin A$. Now by injectivity of f, $f(z) \notin f(A)$ or $f(z) \in f(X) - f(A)$, i.e., $f(X - A) \subseteq f(X) - f(A)$.
(g) Observe that $x \in f^{-1}(Y - P)$ if and only if $f(x) \in (Y - P)$ if and only if $f(x) \notin P$ if and only if $x \notin f^{-1}(P)$ if and only if $x \in X - f^{-1}(P)$. So, the assertion follows. ■

Example. Let $f : \{1,2,3\} \to \{1,2,3\}$, such that $f(1) = 1 = f(3)$ and $f(2) = 2$. Take $A = \{1,2\}$ and $B = \{2,3\}$. Then $f(A \cap B) = \{2\}$, whereas $f(A) \cap f(B) = \{1,2\}$. Thus $f(A \cap B) \neq f(A) \cap f(B)$ in general.

Theorem 3. *There does not exist a surjective map from a set X to $\mathcal{P}(X)$.*

Proof. If possible, let $f : X \to \mathcal{P}(X)$ be surjective. Let $B = \{x \mid x \notin f(x)\}$, such that $B \in \mathcal{P}(X)$. Let $f(z) = B$ for some $z \in X$, since f is surjective. Then $z \in B$ if and only if $z \notin f(z) = B$, which is absurd. So, f does not exist. ■

Definition. Let J be a subset of the set of real numbers and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a family of sets indexed by J, i.e., $A_\alpha \in \mathcal{A}$ for each $\alpha \in J$. The arbitrary union of members of $\mathcal{A}$ denoted $\cup_{\alpha\in J} A_\alpha$ is defined as the smallest set containing A_α for each $\alpha \in J$, i.e., if there is a set B that contains each set A_α, then $B \supseteq \cup_{\alpha\in J} A_\alpha$. More precisely,

$$\cup_{\alpha\in J} A_\alpha := \{x \mid x \in A_\alpha, \text{ for some } \alpha \in J\}. \tag{1.3}$$

Similarly, $\cap_{\alpha\in J} A_\alpha$ is defined as the largest set contained in each A_α, i.e.,

$$\cap_{\alpha\in J} A_\alpha := \{x \mid x \in A_\alpha, \text{ for every } \alpha \in J\}. \tag{1.4}$$

Example. Let $\{A_\alpha\}_{\alpha\in J}$ be an indexed family of subsets of a set X. Then the following version of one of the De Morgan's two laws is true: $X - \cup_{\alpha\in J} A_\alpha = \cap_{\alpha\in J}(X - A_\alpha)$, which follows from the following steps:
An element $x \in X - \cup_{\alpha\in J} A_\alpha \Leftrightarrow x \notin \cup_{\alpha\in J} A_\alpha \Leftrightarrow x \notin A_\alpha$ for each $\alpha \in J \Leftrightarrow x \in (X - A_\alpha)$ for each $\alpha \in J \Leftrightarrow x \in \cap_{\alpha\in J}(X - A_\alpha)$.

Exercises

1. Let X and Y are two sets, $f : X \to Y$. For $A \subseteq X$, prove that $f(X - A) \supseteq Y - f(A)$, and for $P \subseteq Y$, $f^{-1}(Y - P) = X - f^{-1}(P)$.

2. Let X, Y, and Z are three sets. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are two maps, such that $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $(x, y) \in X \times Y$. For any $U \subseteq X$ and $V \subseteq Y$, prove that $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = (U \times Y) \cap (X \times V) = U \times V$.

3. Establish the following version of the De Morgan laws for an indexed family $\{A_\alpha\}_{\alpha \in J}$ of subsets of a set X.

   (a) $X - \cup_{\alpha \in J} A_\alpha = \cap_{\alpha \in J}(X - A_\alpha)$

   (b) $X - \cap_{\alpha \in J} A_\alpha = \cup_{\alpha \in J}(X - A_\alpha)$

4. For a nonempty set J, let $\{A_\alpha\}_{\alpha \in J}$ be a collection of subsets of a set X and Y is any other set. If $f : X \to Y$, prove that $f(\cup_{\alpha \in J} A_\alpha) = \cup_{\alpha \in J} f(A_\alpha)$ and $f(\cap_{\alpha \in J} A_\alpha) \subseteq \cap_{\alpha \in J} f(A_\alpha)$.

1.2 Relations

Definition. A relation from a set A to a set B is a subset of $A \times B$. If R is a relation from a set A to a set B, i.e., $R \subseteq A \times B$ and $(x, y) \in R$, we express this as xRy, and say that, x is related to y via R.

Example. A function $f : A \to B$ can be regarded as a relation f from A to B, such that for each $x \in A$,
(i) xfy(to mean $y = f(x)$) for some $y \in B$
(ii) if for $a \in A$, afb and afc for some $b, c \in B$, then $b = c$

1.2.1 Equivalence relations

Definition. A relation $\sim$ on a set $S \neq \emptyset$ is said to be an equivalence relation if it is
(a) *Reflexive*: $x \sim x$ for all $x \in S$
(b) *Symmetric*: If $x \sim y$ in S then $y \sim x$
(c) *Transitive*: If $x \sim y$ and $y \sim z$ in S then $x \sim z$

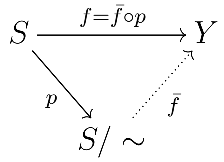
Example. Let $X = \{1, 2, 3\}$. Then the relation $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ is an equivalence relation on X. The relation $R_2 = \{(1, 1), (2, 2), (2, 3), (1, 2)\}$ is not transitive since $(1, 2)$, $(2, 3) \in R_2$ but $(1, 3) \notin R_2$. The relation $R_3 = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$ is not reflexive as $(3, 3) \notin R_3$.

Definition. Let $\sim$ be an equivalence relation on a set S. For any $x \in S$ the equivalence class of x under $\sim$ is defined as the set $\bar{x} := \{y \in S : y \sim x\}$.

Theorem 4. *Any two equivalence classes of an equivalence relation are either disjoint or same.*

Proof. Let $\bar{x}$ and $\bar{y}$ are any two equivalence classes of an equivalence relation $\sim$ on a set $S \neq \emptyset$. If $\bar{x} \neq \bar{y}$, suppose that $\bar{x} \cap \bar{y} \neq \emptyset$. So, let $z \in \bar{x} \cap \bar{y}$. Then $z \in \bar{x}$, which gives $z \sim x$ and $z \in \bar{y}$; so that $z \sim y$. By symmetry, $y \sim z$. As $y \sim z$ and $z \sim x$, by transitivity, $y \sim x$ or $y \in \bar{x}$. Now for each $a \in \bar{y}$, $a \sim y$ and as $y \sim x$, by transitivity, $a \sim x$. Thus, $\bar{y} \subseteq \bar{x}$. Interchanging the role of x and y will prove $\bar{x} \subseteq \bar{y}$. ■

Theorem 5. *Denote by $S/\sim$, the set of all equivalence classes of an equivalence relation $\sim$ on a set S. Then the map $p : S \to S/\sim$ defined by $p(x) = \bar{x}$ is surjective. If $f : S \to Y$ is another function such that whenever $x \sim y$ in S gives $f(x) = f(y)$ then there is a map$\bar{f} : S/\sim \to Y$ for which $f = \bar{f} \circ p$, i.e., the following diagram commutes:*



Proof. Clearly p is surjective. Define $\bar{f}(\bar{x}) = f(x)$ for all $\bar{x} \in S/\sim$. We will establish that $\bar{f}$ is well-defined. So, let $\bar{x} = \bar{y}$. Then $x \sim y$, which gives $f(x) = f(y)$ or $\bar{f}(\bar{x}) = \bar{f}(\bar{y})$. Finally, we have $f(x) = \bar{f}(\bar{x}) = (\bar{f} \circ p)(x)$ for all $x \in S$, which proves $f = \bar{f} \circ p$. ■

Example. Consider $\sim$ on $\mathbb{R}$ as follows: $x \sim y$ if $x - y$ is an integer. It is easy to verify that $\sim$ is an equivalence relation on $\mathbb{R}$, such that $\mathbb{R}/\sim = \{\bar{x} \mid x \in [0,1)\}$. Define $f : \mathbb{R} \to S^1$ by $f(t) := (\cos(2\pi t), \sin(2\pi t))$. If $x \sim y$, then $x - y = n$ for some integer n. Then $f(x) = (\cos(2\pi(n+y)), \sin(2\pi(n+y))) = f(y)$. Also, f is surjective. By the preceding theorem, f factors as $f = \bar{f} \circ p$, where $\bar{f}(\bar{x}) = f(x)$. Note that $\bar{f}$ is a bijection.

Definition. Let $S \neq \emptyset$. A partition π of S is a set of pairwise disjoint subsets of S, such that union of all members of π equals S.

Theorem 6. *A set π consisting of subsets of a set S, such that $\cup_{A \in \pi} = S$ is a partition of S if and only if each member of π is an equivalence class of some equivalence relation on S.*

Proof. If each member of π is an equivalence class of some equivalence relation $\sim$ on S, and it is given that $\cup_{A \in \pi} = S$; by theorem 4, π is a partition of S.
Conversely, let π be a partition of S. Define a relation $\sim$ on S by $x \sim y$ if $x, y \in C$ for some $C \in \pi$. It is easy to verify that $\sim$ is an equivalence relation. Define the surjective map $f : S \to \pi$, such that $f(a) = C$ if $a \in C$ for some $C \in \pi$. By Theorem 5, the function f factors into $\bar{f} \circ p$, where $\bar{f} : S/\sim \to \pi$ is also surjective. We only need to show that $\bar{f}$ is injective. For this, let $\bar{f}(\bar{x}) = \bar{f}(\bar{y})$, then $x, y \in C$ for some $C \in \pi$. So, $x \sim y$ or $\bar{x} = \bar{y}$. ■

1.2.2 Order relations

Definition. Let X be a nonempty set. A relation $\prec$ on X is called an order-relation or simple-order if for all $x, y, z \in X$, the following hold
(a) *Comparability*: for all $x \neq y$, either $x \prec y$ or $y \prec x$
(b) *Non-reflexivity*: $x \not\prec x$ for all x
(c) *Transitivity*: if $x \prec y$, and $y \prec z$ then $x \prec z$
The set X with an order relation is called a simply ordered set.

A subset A of a simply ordered set X is simply ordered with respect the relation of X restricted to A.

Example. Let X be a nonemptyset. Define the relation $\prec$ on the set $\mathcal{S}$ consisting of those subsets of X, which are ordered by strict set inclusion $\subsetneq$. Comparability holds in the definition of $\mathcal{S}$. For all $A \in \mathcal{S}$, $A \not\subsetneq A$, which gives $A \not\prec A$. So, the relation $\prec$ is non-reflexive. To verify transitivity, assume that $A \prec B$ and $B \prec C$. Then $A \subsetneq B$ and $B \subsetneq C$, which imply $A \subsetneq C$ or $A \prec C$. Thus, $\prec$ is a simple order on $\mathcal{S}$.

Example. The *usual order* $<$ on the set of all real numbers $\mathbb{R}$ is a simple order on $\mathbb{R}$.

Definition. A strict partial order denoted on a set X is a relation on X, which is non-reflexive and transitive.

Example. A simple order is also a strict partial order.

Example. Let X be a set with at least two elements. Define a relation $\prec$ on $\mathcal{P}(X)$ by $A \prec B$ if $A \subsetneq B$. Clearly, $\prec$ is non-reflexive and transitive. So, $\prec$ is a strict partial order on $\mathcal{P}(X)$. Note however that, $\prec$ is not a simple order on $\mathcal{P}(X)$, since comparability does not always hold.

Definition. Let $\prec$ be a strict partial order on a set X. A partial order denoted $\preceq$ on X is the one which satisfies the following axioms
(a) *Reflexivity*: $x \preceq x$ for all $x \in X$
(b) *Antisymmetry*: if $x \preceq y$ and $y \preceq x$ then $x = y$
(c) *Transitivity*: if $x \preceq y$ and $y \preceq z$ then $x \preceq z$

Example. The *usual partial order* $\leq$ of $\mathbb{R}$ defined by $x \leq y$ to mean 'either $x < y$ or $x = y$' is a partial order on $\mathbb{R}$.

1.3 Least upper bound property

Definition. A set X with a simple order $\prec$ is called an ordered set. A subset S of X is also an ordered set with the restriction of the order of X to S. $S \subseteq X$ is said to be bounded above in X if there is some $x \in X$, such that for all $s \in S$, either $s = x$ or $s \prec x$. The element x (whenever it exists) is called an upper bound of S in X. Similarly, a set $T \subseteq X$ is said to be bounded below in X, if there is a $y \in X$, such that either $y = t$ or $y \prec t$ for all $t \in T$. The element y (whenever it exists) is called a lower bound of T in X.

Definition. Let X be an ordered set with order relation $\prec$, and let $S \subseteq X$ is bounded above in X. An element $\alpha \in X$ is said to be the least upper bound (lub) or supremum of S in X, if
(i) α is an upper bound of S
(ii) if for any $\beta \in X$, $\beta \prec \alpha$ then β is not an upper bound of S
If $\alpha \in X$ is the least upper bound of S in X, we write $\alpha = \sup S$.
Similarly, if a subset T of X is bounded below in X, an element $\mu \in X$ is said to be the greatest lower bound (glb) or infemum of T in X if
(i) μ is a lower bound of T
(ii) if for any $\lambda \in X$, $\mu \prec \lambda$ then λ is not a lower bound of T.
If μ is glb of T, we write $\mu = \inf T$.

Definition. Let A be a strict partially ordered set. An element $\gamma \in A$ is said to be a maximal element of A, if there does not exist any element $x \in A$, such that $\gamma \prec x$. Minimal element of A is defined analogously.

Example. Consider the ordered set of positive integers $\mathbb{Z}_+ \subseteq \mathbb{R}$ with the simple order as restriction of the usual order $<$ of $\mathbb{R}$. Then $-1, 1 \in \mathbb{R}$, such that $-1 < x$ and $1 \leq x$ for all $x \in \mathbb{Z}_+$. So, -1 and 1 are lower bounds of the set $\mathbb{Z}_+$ in $\mathbb{R}$. In fact every real number α, such that $\alpha < 1$ or $\alpha = 1$, is a lower bound of $\mathbb{Z}_+$. Here, $\inf\{\mathbb{Z}_+\} = 1$.

Definition. (Interval) Let X be an ordered set with the simple order $\prec$. Then for $a, b \in X$, an open interval (a, b) in X is defined as the set $(a, b) := \{x \in X \mid a \prec x \text{ and } x \prec b\}$. The set $[a, b] := (a, b) \cup \{a, b\}$ is called the closed interval in X.

Example. For $a,\ b \in \mathbb{R}$, such that $a < b$, the set $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ is an open interval.

Example. Under the usual order of $\mathbb{R}$, the open interval $(0, 1)$ does not have any maximal and minimal elements. Let us see how? If possible, let α is a maximal element of $(0, 1)$. Then $\alpha < 1$ and $\alpha < \frac{\alpha+1}{2} < 1$, such that $\frac{\alpha+1}{2} \in (0, 1)$, which contradicts the definition of maximal element. Similarly, one can establish that $(0, 1)$ does not have any smallest element. Check that $\sup\{(0, 1)\} = 1$ and $\inf\{(0, 1)\} = 0$.

We now define one of the fundamental properties of the real line namely, *the least upper bound property*, which will serve as a basis to establish some of the important basic results from analysis.

Definition. An ordered set X is said to have the least upper bound property if for every *nonempty* subset S of X, which is bounded above in X, $\sup\{S\}$ exists in X.

Example. The ordered field of real numbers $\mathbb{R}$ has the least upper bound property.

Example. The field of rational numbers $\mathbb{Q}$ does not have the least upper bound property; because, the set $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is a subset of $\mathbb{Q}$, which has the least upper bound $\sqrt{2} \notin \mathbb{Q}$.

1.3.1 The real line

Axiom 1.(Existence of real line) *There exists an ordered field* $(\mathbb{R}, <)$ *having the least upper bound property, such that it contains a copy of* $\mathbb{Q}$ *as a proper subfield.*

We will denote the order on $\mathbb{R}$ by $<$ with respect to which it satisfies the least upper bound property and call the ordered pair $(\mathbb{R}, <)$ *the real line*. The order $<$ is called the usual order of $\mathbb{R}$. We usually omit the symbol $<$ and use the symbol $\mathbb{R}$ to denote the real line. Each element of the ordered field $\mathbb{R}$ is called *a real number*. For a construction of the ordered field of real numbers, the reader may consult Rudin's book [1]. Here we take the existence of such an ordered field as an axiom and build upon the defining properties of the real numbers.

We list the axioms satisfied by the real numbers, which are the elements of $\mathbb{R}$ with respect to the commutative binary operations of addition '+', multiplication '$\cdot$', & the usual order '$<$'.

Algebraic properties

1. $x + (y + z) = (x + y) + z$; $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \mathbb{R}$.
2. $x + y = y + x$; $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}$.
3. There is a unique real numbers 0 and 1 satisfying

   $x + 0 = 0 + x = x$ for all $x \in \mathbb{R}$ and $x \cdot 1 = 1 \cdot x = x$ for all $x \in \mathbb{R} - \{0\}$.
4. For each $x \in \mathbb{R}$, there is unique $y \in \mathbb{R}$ such that $x+y = y+x = 0$. For each real number $x \neq 0$, there is unique real number $y \neq 0$, such that $x \cdot y = y \cdot x = 1$.
5. $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.

Mixed algebraic and order properties

6. If $x + y < x + z$ then $y < z$;

   if $x < y$ and $0 < z$ then $x \cdot z < y \cdot z$.

Order properties (Topological!)

7. The simple order $<$ has least upper bound property.
8. If $x < y$ then there is a $z \in \mathbb{R}$, such that $x < z$ and $z < y$.

1.4 Maximum principle and Zorn's lemma

To go further, we need a couple of axioms from the set-theory, so let us state them first.

Axiom 2.(axiom of choice) *For a given collection $\mathcal{A}$ of pairwise-disjoint nonempty sets, there exists a set $\mathcal{C}$ such that for every $A \in \mathcal{A}$, $A \cap \mathcal{C}$ is a singleton set.*

The axiom of choice was formulated by Ernst Zermelo in 1904 for proving the well-ordering theorem.

Proposition 7. *Let $\mathcal{B}$ be a collection of pairwise-disjoint nonempty sets. Then there exists a choice function $f : \mathcal{B} \to \cup_{B \in \mathcal{B}} B$ such that $f(B) \in B$ for all $B \in \mathcal{B}$.*

Proof. By axiom of choice, there is a set $\mathcal{C}$ such that $B \cap \mathcal{C}$ is singleton for each $B \in \mathcal{B}$. Define $f(B)$ to be the element of the set $B \cap \mathcal{C}$. ■

The next result shows that a choice function on any collection of nonempty sets to their union can also be constructed.

Theorem 8. *Let $\mathcal{B}$ be a collection of nonempty sets not necessarily disjoint. Then there exists a choice function $f : \mathcal{B} \to \cup_{B \in \mathcal{B}} B$ such that $f(B) \in B$ for all $B \in \mathcal{B}$.*

Proof. Consider the subset $\mathcal{A}$ of the cartesian product $\mathcal{B} \times \cup_{B \in \mathcal{B}} B$, such that $\mathcal{A} = \{B' \mid B' = \{(B, x) \mid x \in B\},\ B \in \mathcal{B}\}$. Then each $B' \in \mathcal{A}$ is nonempty since so is each $B \in \mathcal{B}$. Also, for any two distinct elements $B'_1,\ B'_2$ of $\mathcal{A}$, such that $(B_1, x) \in B'_1$ and $(B_2, y) \in B'_2$, we have $(B_1, x) \neq (B_2, y)$ since $B_1 \neq B_2$. Thus B'_1 and B'_2 are disjoint. We have proved that $\mathcal{A}$ is a collection of pairwise-disjoint nonempty sets. By axiom of choice, there is a set $\mathcal{C}$ such that for each member B' of $\mathcal{A}$, $B' \cap \mathcal{C}$ is singleton set of the form (B, x), where $x \in B \in \mathcal{B}$. Define f to be the rule that associates the first coordinate of $B' \cap \mathcal{C}$ to its second coordinate, i.e., $f(B) = x \in B$. ■

Axiom 3.(Maximum principle) *Every strict partially ordered set has a maximal simply ordered subset.*

Axiom 4.(Zorn's lemma) *Let S be a nonempty set with strict partial order. If every simply ordered subset of S has an upper bound in S then S itself has a maximal element.*

It turns out that the axiom of choice, the maximum principle, and the Zorn's lemma are equivalent. However here, we prove equivalence of maximum principle and the Zorn's lemma.

Theorem 9. *Maximum principle implies Zorn's lemma and vice versa.*

Proof. First we derive Zorn's lemma from maximum principle. Let A be a set with strict partial order $\prec$, such that every simply ordered subset of A has an upper bound in A. Using maximum principle, let $C \subseteq A$ is a maximal simply ordered subset. By hypothesis, let α be an upper bound of C in A. Then α is a maximal element of A, because, if $\alpha \prec d$ for some $d \in A$ then $c \prec d$ for all $c \in C$. So, the set $C \cup \{d\}$ is also a simply ordered subset of A properly containing C, which contradicts maximality of C and proves Zorn's lemma.

Conversely, let A be a set with strict partial order $\prec$. Consider the collection $\mathcal{C}$ of all subsets of A which are simply ordered by $\prec$. Consider the set $\mathcal{C}$ with proper set inclusion $\subsetneq$ as a strict partial order. If $\mathcal{D}$ is a simply ordered subset of $\mathcal{C}$ then $\cup_{D\in\mathcal{D}}D$ is an upper bound of $\mathcal{D}$. We need to show that $\cup_{D\in\mathcal{D}}D \in \mathcal{C}$, i.e., $\cup_{D\in\mathcal{D}}D$ is simply ordered with respect to the order relation $\prec$. We only need to establish comparability. So, pick two distinct elements x and y from $\cup_{D\in\mathcal{D}}D$. Then there exist $D_1 \ni x$ and $D_2 \ni y$ in $\mathcal{D}$. Since the set $\mathcal{D}$ is simply ordered by set inclusion, either $D_1 \subsetneq D_2$ or $D_2 \subsetneq D_1$. Since D_1 and D_2 are simply ordered by $\prec$, either $x \prec y$ or $y \prec x$, which proves that $\cup_{D\in\mathcal{D}}D \in \mathcal{C}$. As $\mathcal{D}$ is arbitrary, it follows that every simply ordered subset of $\mathcal{C}$ has an upper bound in $\mathcal{C}$. By Zorn's lemma, $\mathcal{C}$ has a maximal element, which is the maximal simply ordered subset of A. ■

Example. Here we give a nice application of Zorn's Lemma in linear algebra, namely, the following. *Every vector space has a basis.*

Proof. Let V be a vector space over a field F. We know that if $S \subseteq V$ then span$S$ consists of all finite linear combinations of the elements of S. Now consider the collection $\mathcal{C}$ of all linearly independent subsets of V with strict partial order defined by the proper set inclusion. Note that if $B \in \mathcal{C}$ then every subset of B is also in $\mathcal{C}$ since every subset of a linearly independent set is linearly independent. If $\mathcal{D}$ be a simply ordered subset of $\mathcal{C}$ then observe that $\cup_{D\in\mathcal{D}}D \in \mathcal{C}$. Hence by Zorn's Lemma, $\mathcal{C}$ itself has a maximal element $\mathscr{B}$. Since $\mathscr{B}$ is linearly independent and no other such subset of V contains it properly, Span$\mathscr{B} = V$ otherwise if $v \in V -$ Span$\mathscr{B}$ then $\mathscr{B} \cup \{v\}$ is linearly independent and contains $\mathscr{B}$ properly thereby contradicting maximality of $\mathscr{B}$. ■

Example.

1.5 Positive integers and strong induction principle

Since we will often use induction principle for proving some of the results in topology, we need to define positive integers first and state and prove the induction principle as well.

Definition. A subset A of $\mathbb{R}$ is called an inductive set if $1 \in A$, and $1 + x \in A$, whenever $x \in A$. The minimal inductive set, denoted $\mathbb{Z}_+$, is called the set of positive integers, i.e.,

$$\mathbb{Z}_+ := \cap_{A \text{ inductive}} A. \tag{1.5}$$

Verify that $\mathbb{Z}_+ \neq \emptyset$ and satisfies the least upper bound property (see Theorem 12). The set of integers is denoted $\mathbb{Z}$ and it is defined as the union of $\mathbb{Z}_+$, the zero element of $\mathbb{R}$, and the set $\mathbb{Z}_-$ consisting of additive inverses of the elements of $\mathbb{Z}_+$. Thus,

$$\mathbb{Z} := \mathbb{Z}_- \cup \{0\} \cup \mathbb{Z}_+. \tag{1.6}$$

Definition. The set of *quotients* of integers denoted $\mathbb{Q}$ is called the set of *rational numbers.*

Definition. For each $n \in \mathbb{Z}_+$ define the set $S_{n+1} = \{1, 2, \ldots, n\}$.

Theorem 10. (Well ordering property) *Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.*

Proof. We first establish that every nonempty subset of the set S_{n+1} has a smallest element. Let A be the set, such that $n \in A$ if every subset of S_{n+1} has a smallest element. Clearly, $1 \in A$ since S_2 is the singleton set $\{1\}$. So, 1 itself is the smallest element of S_2. Now suppose that $m \in A$. We show that $m+1 \in A$. To do so, let C be a nonempty subset of S_{m+2}. If $C = \{m+1\}$ then $m+1$ is the smallest element of C; otherwise $(C - \{m+1\}) \cap S_{m+1}$ is nonempty. Since $m \in A$, $C - \{m+1\}$ has a smallest element say α. Then no member of $S_{m+1} \cap C$ is smaller than α and $\alpha < m+1$. Thus, C has α as a smallest element. Since C is arbitrary, every subset of S_{m+2} has a smallest element. Hence $m+1 \in A$. We have shown that A is inductive; therefore, $A = \mathbb{Z}_+$, and the assertion follows.
Now we prove the theorem. Let B be any nonempty subset of $\mathbb{Z}_+$. Choose an element n of B. Then $A = B \cap S_{n+1}$ is nonempty, so that A has a smallest element s. We claim that s is the smallest element of B, otherwise if there is a $x \in B$ such that $x < s$, then $x \in S_{n+1}$. Thus $x \in (B \cap S_{n+1}) = A$, which contradicts the fact that s is the smallest element of A. ■

Theorem 11. (Strong induction principle) *Let A be a set of positive integers. Suppose that for each positive integer n, the statement $S_n \subseteq A$ implies the statement $n \in A$. Then $A = \mathbb{Z}_+$.*

Proof. If possible, let $A \neq \mathbb{Z}_+$, and let n is the smallest positive integer that is not in A. Then for every positive integer $m < n$, $m \in A$. So, by hypothesis $n \in A$, which contradicts the assumption. ■

Theorem 12. *A well ordered set has the least upper bound property.*

Proof. Let X be a well ordered set such that $S \subseteq X$ has an upper bound in X. Let B be the set of all upper bounds of S. By well ordering of X, B has a smallest element in X. Denote the smallest element of B by α. We have shown that there is no upper bound of S smaller than α, i.e., $\sup S = \alpha \in X$. ■

Corollary 13. *The set of all positive integers $\mathbb{Z}_+$ has the least upper bound property with respect to the usual ordering of real numbers.*

Theorem 14. *Any simply ordered set having the least upper bound property has the greatest lower bound property.*

Proof. Let X be a set, which is simply ordered by relation $\prec$, and X has least upper bound property. Let G be a subset of X, which is bounded below in X. Consider the set L of all lower bounds of G in X. For every $\ell \in L$, and $g \in G$, either $\ell \prec g$ or $\ell = g$. So, L is bounded above in X by each element of G. By least upper bound property of X, let $\sup L = \alpha \in X$. We only need to prove that α is a lower bound of G. Suppose not, then there is a $g \in G$, such that $g \prec \alpha$. Then g is an upper bound of L smaller than α, which is a contradiction. ∎

1.6 Countable and uncountable sets

Definition. We say that the cardinal number or cardinality of a A is less than or equal to the cardinal number of a set B if either $A = \emptyset$ or there exists an injective map from A to B. We express this relation by writing $|A| \leq |B|$. The sets A and B are said to have same cardinal number if either $A = B = \emptyset$ or there is a bijection between them, and we write $|A| = |B|$. Cardinal number of empty set is defined to be zero.

Definition. Let X and Y be two nonempty sets, and let $f : X \to Y$ is a map. For any $A \subseteq X$, the restriction of the map f to the subset A is the map $g : A \to Y$, such that $g(a) = f(a)$ for all $a \in A$, and we write $g = f|_A$.

Proposition 15. *If $A \subseteq B$ then $|A| \leq |B|$.*

Proof. If $A = \emptyset$ then we are done. If $A \neq \emptyset$ then the map $Id_B|_A$ is injective; So, $|A| \leq |B|$. ∎

Theorem 16. (Schröeder-Berstein) *If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

Proof. If $|A| \leq |B|$ and $|B| \leq |A|$ then there exist $f : A \to B$ and $g : B \to A$, such that f and g are injective. Define the following sets:

$$Y_0 = B - f(A);\ Y_n = (f \circ g)^n(Y_0),\ Z_{n-1} = g(Y_{n-1}),\ n \in \mathbb{Z}_+. \tag{1.7}$$

Now let $Y = \cup_{n=0}^{\infty} Y_n$ and $Z = \cup_{n=0}^{\infty} Z_n$. Note that $g : Y_n \to Z_n$ is bijective, and so is $g : Y \to Z$. Define $F : A \to B$, such that

$$F(x) = \begin{cases} g^{-1}(x), & \text{if } x \in Z; \\ f(x), & \text{if } x \in A - Z. \end{cases} \tag{1.8}$$

Claim I: $z \in Z$ if and only if $f(z) \in Y$. If $z \in Z$ then $z = g(y)$, where $y \in Y_n$ for some $n \geq 0$. Then $f(z) = (f \circ g)(y) \in (f \circ g)(Y_n) = Y_{n+1}$. Thus, $f(z) \in Y$. On the other hand if $f(z) \in Y$ then $f(z) = (f \circ g)^m(s)$ for some $s \in Y_0$ and $m \geq 1$. Since f is injective, $z = g(f \circ g)^{m-1}(s) \in Z_{m-1}$. Thus, $z \in Z$ and the claim is proved.

Claim II: F is injective. Let $F(x_1) = F(x_2)$, for $x_1,\ x_2 \in X$. If $x_1 \in Z$ and $x_2 \in (A - Z)$ then $g^{-1}(x_1) = f(x_2)$ implies $x_1 = g(f(x_2))$. By Claim I above, $f(x_1) = (f \circ g)(f(x_2)) \in Y$ or $f(x_2) \in Y$ if and only if $x_2 \in Z$ which is a contradiction. Hence $x_2 \in Z$, and $F(x_1) = F(x_2)$ implies $g^{-1}(x_1) = g^{-1}(x_2)$ or $x_1 = x_1$, since g is injective.

Claim III: F is surjective. Let $b \in B$. If $b \in Y$, then $b = g^{-1}(a) = F(a)$ for some $a \in Z$, and we are done. If $b \in B - Y$ then $b \notin Y_0 = B - f(A)$ or $b \in f(A)$. So, $b = f(c) = F(c)$ for some $c \in A$, which proves that F is surjective. ■

Example. Let $f : (0, 1) \to [0, 1)$ and $g : [0, 1) \to (0, 1)$ be defined as $f(x) = x$ and $g(x) = \dfrac{1 + x}{2}$. Clearly, f and g are injective. Using the construction in the preceding proof, we have $(f \circ g)^n(x) = 1 + \dfrac{x - 1}{2^n}$, $Y = \{1 - \dfrac{1}{2^n} \mid n = 0, 1, \ldots\}$, and $Z = g(Y) = \{1 - \dfrac{1}{2^{n+1}} \mid n = 0, 1, \ldots\}$. Thus, the required bijection is $F : (0, 1) \to [0, 1)$, such that

$$F(x) = \begin{cases} 1 - \dfrac{1}{2^n}, & \text{if } x = 1 - \dfrac{1}{2^{n+1}}; \\ x, & \text{if } x \neq 1 - \dfrac{1}{2^{n+1}}. \end{cases} \tag{1.9}$$

for each $n = 0, 1, \ldots$.

Definition. Call a set A a *finite set* if either $A = \emptyset$ or there exists a positive integer n and a bijection between A and the set S_{n+1}. The number n is called the cardinal number of A, i.e., $|A| = n$, and we say that the set A has n elements.

Example. The set S_{n+1} is finite for every n, since the map $Id_A : A = S_{n+1} \to S_{n+1}$ defined by $Id_A(x) = x$ is bijective.

Example. Let B be a finite set, and let $A \subseteq B$. Then $|A| \leq |B|$, so A is also finite. Thus every finite subset of a finite set is finite.

Theorem 17. (Pigeon-hole principle) *Let A and B are finite sets such that $|A| = |B|$. Then any injective map $f : A \to B$ is surjective.*

Proof. Let $|A| = n$. We will prove the result using induction on n. If $n = 1$ then $f(A) = B$, and f is surjective. Suppose that the result is true for all positive integers $n \leq k$ for some $k \in \mathbb{Z}_+$. Let $n = k + 1$. Take any $a \in A$. As $f : A - \{a\} \to B - \{f(a)\}$ is injective, and $|A - \{a\}| = |B - \{f(a)\}| = k < k + 1$, by hypothesis, $f : A - \{a\} \to B - \{f(a)\}$ is surjective. Thus, the result is true for $n = k + 1$, which completes final step of induction. ■

Example.

Definition. A set is said to be *infinite* if it is not finite.

Proposition 18. *The set $\mathbb{Z}_+$ is an infinite set.*

Proof. If possible, let $\mathbb{Z}_+$ is finite, and $f : \mathbb{Z}_+ \to S_{n+1}$ is a bijection for some $n \in \mathbb{Z}_+$. Then the restriction $f|_{S_{n+2}} : S_{n+2} \to S_{n+1}$ is injective. So, $n+1 = |S_{n+2}| \leq |S_{n+1}| = n$ or $1 \leq 0$, which is absurd. Thus, the set $\mathbb{Z}_+$ must be infinite. ■

Theorem 19. *A set X is infinite if and only if there is an injective map $f : \mathbb{Z}_+ \to X$.*

Proof. Let X is an infinite set. Let $x_1 \in X$. Using axiom of choice, let $c : \mathcal{P}(X) - \emptyset \to X$ be the choice function satisfying $x_1 = c(X)$, and $x_{n+1} = c(X - \{x_1, \ldots, x_n\})$ for all $n \in \mathbb{Z}_+$, which is well defined since $(X - \{x_1, \ldots, x_n\}) \neq \emptyset$ as X is infinite. The required injective map is given by $n \mapsto x_n$.
Conversely, **(NOT $Q \Rightarrow$ NOT P)** if X is finite then there is no injective map $f : \mathbb{Z}_+ \to X$. ■

Remark. The logical statement **'**$P \Rightarrow Q$**'** is equivalent to the statement **'NOT** $Q \Rightarrow$ **NOT** P**'**. So, proving one of these statements proves the other one.
For example, In the preceding proof, the statements 'if there is an injective map $f : \mathbb{Z}_+ \to X$ then X is infinite' and 'if X is finite then there is no injective map $f : \mathbb{Z}_+ \to X$' are logically equivalent.

Definition. (Countable set) A set X is said to be countable if either X is finite or there is a bijection $f : \mathbb{Z}_+ \to X$.

From the above definition and Theorem 19, it can be deduced that a set X is countable if and only if $|X| \leq |\mathbb{Z}_+|$, where the equality holds if X is infinite. Thus, the set X is countable if and only if there is an injective map $f : X \to \mathbb{Z}_+$.

Example. Every finite set is countable

Example. The set of all positive integers $\mathbb{Z}_+$ is countable, since the map $n \mapsto n$ is a bijection of $\mathbb{Z}_+$.

Example. A subset of a countable set is countable, since restriction of a bijection on to its image is also a bijection.

Example. The map $f : \mathbb{Z}_+ \to \mathbb{Z}_+ \times \mathbb{Z}_+$ defined by $f(n) = (a, b)$, where the positive integers a and b satisfy $n+1 = 2^a(2b+1)$, is a bijection. Thus, $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable.

Example. The set of all integers $\mathbb{Z}$ is countable, since the map $f : \mathbb{Z}_+ \to \mathbb{Z}$ defined by $f(n) = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd;} \\ -\frac{n}{2}, & \text{if } n \text{ is even,} \end{cases}$ is a bijection.

Theorem 20. *Following results are true regarding countable sets:*
(a) Countable union of countable sets is countable.
(b) Cartesian product of two countable sets is countable.

Proof. (a) Let $\{A_n\}_{n\in\mathbb{Z}_+}$ be a countable family of countable sets. For each n, choose a bijection $f_n : A_n \to \mathbb{Z}_+$, which exists since A_n is countable. Define $f : \cup_{n=1}^{\infty} A_n \times \{n\} \to \mathbb{Z}_+$, such that whenever $k \in A_n$, $f(k,n) = p_n^{f_n(k)}$, where p_k is the k-th prime number. If $f(k_1,n_1) = f(k_2,n_2)$ then $p_{n_1}^{f_{n_1}(k_1)} = p_{n_2}^{f_{n_2}(k_2)}$, which is possible only if $p_{n_1} = p_{n_2}$, since $f_{n_1}(k_1)$ and $f_{n_2}(k_2)$ are positive integers and $f_{n_1}(k_1) = f_{n_2}(k_2)$. Thus $n_1 = n_2$ so that $f_{n_1}(k_1) = f_{n_1}(k_2)$ which implies $k_1 = k_2$, since f_{n_1} is a bijection. Thus, $(n_1,k_1) = (n_2,k_2)$ and f is injective.
Observe that $F : \mathbb{Z}_+ \to \cup_{n=1}^{\infty} A_n \times \{n\}$ such that $F(n) = (f_1^{-1}(n), n)$ is injective. is injective. Therefore, by Schröeder-Bernstein theorem, there is a bijection $h : \mathbb{Z}_+ \to \cup_{n=1}^{\infty} A_n \times \{n\}$. Since the map $F : \cup_{n=1}^{\infty} A_n \times \{n\} \to \cup_{n=1}^{\infty} A_n$ defined by $F(x,n) = x$, whenever $x \in A_n$ is bijective, so is the composition $(F \circ h) : \mathbb{Z}_+ \to \cup_{n=1}^{\infty} A_n$.
(b) Let A and B are two countable sets. So, let $f_A : \mathbb{Z}_+ \to A$ and $f_B : \mathbb{Z}_+ \to B$ be bijective maps. Then the map $f : \mathbb{Z}_+ \to A \times B$ defined by $f(n) = (f_A(n), f_B(n))$ is injective. Also, the map $g : A \times B \to \mathbb{Z}_+$, such that $g(a,b) = 2^{f_A^{-1}(a)} 3^{f_B^{-1}(b)}$ is injective. Again by Schröeder-Bernstein theorem, there is a bijection between $\mathbb{Z}_+$ and $A \times B$. Thus, $A \times B$ is countable. ∎

Example. Let $\mathbb{Q}_+$ denotes the set of all positive rational numbers. Then the maps $f : \mathbb{Z}_+ \to \mathbb{Q}$, such that $f(n) = n$, and $g : \mathbb{Q}_+ \to \mathbb{Z}_+$ defined by $g(p/q) = 2^{p/d} 3^{q/d}$, where $d = \gcd(p,q)$ are injective. So, by Schröeder-Bernstein theorem, there is a bijection between $\mathbb{Z}_+$ and $\mathbb{Q}_+$, and $\mathbb{Q}_+$ is countable. Let $\mathbb{Q}_- = \{-r \mid r \in \mathbb{Q}_+\}$. Then the map $r \mapsto -r$ from $\mathbb{Q}_+$ to $\mathbb{Q}_-$ is a bijection. Therefore, $\mathbb{Q}_-$ is countable. Finally, the set all rational numbers $\mathbb{Q}$ is countable as it is equal to finite union of countable sets, i.e., $\mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\}$.

Definition. (Uncountable set) A set X is said to be uncountable if X is not countable.

Theorem 21. *Let $\{0,1\}^{\mathbb{N}}$ be the set of all map $f : \mathbb{Z}_+ \to \{0,1\}$. Then the set $\{0,1\}^{\mathbb{N}}$ is uncountable.*

Proof. If possible, let the set $\{0,1\}^{\mathbb{N}}$ is countable. So, let $F : \mathbb{Z}_+ \to \{0,1\}^{\mathbb{N}}$ be a bijection, such that $F(n) = f_n \in \{0,1\}^{\mathbb{N}}$, where $f_n(k) \in \{0,1\}$ for all k. Define $\sigma : \mathbb{Z}_+ \to \{0,1\}$, such that for each positive integer n, $\sigma(n) = 0$ if $f_n(n) = 1$ otherwise $\sigma(n) = 1$. Then $\sigma \in \{0,1\}^{\mathbb{N}}$, such that $\sigma(n) \neq f_n(n) = F(n)$ for all n, which contradicts surjectivity of F. Thus, our supposition is wrong and the set $\{0,1\}^{\mathbb{N}}$ must be uncountable. ∎

Corollary 22. *The set of all real numbers $\mathbb{R}$ is uncountable.*

Proof. Since every real number r has unique binary number representation, i.e., $r = \sum_{n\geq 1} a_r(n)2^{n-1} + \sum_{n\geq 1} b_r(n)2^{-n}$, where $a_r(n),\ b_r(n) \in \{0,1\}$, so that $a_r, b_r \in \{0,1\}^{\mathbb{N}}$, the map $F : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$, such that $F(f) = \sum_{n\geq 1} f(n)2^{n-1}$ is injective. So, $|\{0,1\}^{\mathbb{N}}| \leq |\mathbb{R}|$, where the set $\{0,1\}^{\mathbb{N}}$ is uncountable. Thus, so is $\mathbb{R}$, otherwise, there is an injective map from the set $\{0,1\}^{\mathbb{N}}$ to the set $\mathbb{Z}_+$, such that the set $\{0,1\}^{\mathbb{N}}$ has cardinality of a countable set, which is a contradiction. ∎

Example. The interval $(-\pi/2, \pi/2)$ is uncountable, since the trigonometric function $\tan^{-1} : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection, and $\mathbb{R}$ is uncountable (Corollary 22). Now for any open interval $(a, b),\ a, b \in \mathbb{R}$, the map $F : (a, b) \to (-\pi/2, \pi/2)$, such that

$$F(t) = \pi\Big(\frac{1}{2} - \frac{t-b}{a-b}\Big) \tag{1.10}$$

is a bijection. Thus every open interval in $\mathbb{R}$ is uncountable.

1.7 Basic topology of $\mathbb{R}$

Now we will discussion some of the standard results in the analysis of real-line, which depend upon the order properties (7) and (8) listed on Page 10.

Theorem 23. (Archimedean property) *Let $x,\ y \in \mathbb{R}$, such that $0 < x < y$. Then there exists a positive integer n such that $y < nx$.*

Proof. Let $A = \{nx \mid nx \leq y;\ n \in \mathbb{Z}_+\}$. The set A is bounded above by $y \in \mathbb{R}$. By least upper bound property of $\mathbb{R}$, let $\sup A = \alpha \in \mathbb{R}$. Observe that for each positive integer n, $0 < nx < (n+1)x \leq \alpha$. So, $nx < \alpha$ for all n. In particular, $0 < x < \alpha$ or $0 < \alpha - x < \alpha$, so that $\alpha - x$ is not an upper bound of A. Then there exists a positive integer m, such that $\alpha - x < mx$ or $\alpha < (m+1)x$, which contradicts the fact that $nx < \alpha$ for all n. ∎

Corollary 24. *Let $x,\ y \in \mathbb{R}$, such that $x < y$. Then there exists a $r \in \mathbb{Q}$, such that $x < r < y$.*

Proof. If $x < 0$ and $0 < y$ then $0 \in \mathbb{Q}$, such that $x < 0 < y$, and we are done. Also, if $x,\ y \in \mathbb{Q}$, then $\frac{x+y}{2} \in \mathbb{Q}$, where $x < \frac{x+y}{2} < y$. So, assume that $0 < x < y$, $y \notin \mathbb{Q}$, such that $0 < y - x$. Using Archimedean property of $\mathbb{R}$ (or otherwise) for the pair of positive real numbers $y - x$ and 1, choose a positive integer n such that $1 < n(y - x)$. Then the interval (nx, ny) must contain an integer $m > 0$, and the rational number $\dfrac{m}{n} \in (x, y)$ as required. ∎

Definition. A sequence of real numbers is a map $x : \mathbb{Z}_+ \to \mathbb{R}$. The image set of the sequence x is denoted $\{x_n\}$, where $x(n) = x_n$ for all n. A sequence $\{x_n\} \subseteq \mathbb{R}$ is said to converge to a real number x, if every open interval (a, b) containing x contains all but finitely many terms of the sequence, i.e., there exists a positive integer N, such that $x_n \in (a, b)$ for all $n \geq N$. If $\{x_n\}$ converges to x, we write $\{x_n\} \to x$ or simply $x_n \to x$.

Example. The sequence $\{\frac{1}{n}\}$ converges to 0, since if $(a, b) \ni 0$ choose $\epsilon < b$, such that $0 < \epsilon < 1$. Apply Archimedean property to choose a positive integer N such that $1 < N\epsilon$ or $\frac{1}{N} < \epsilon$. Since $0 < \frac{1}{n} \leq \frac{1}{N} < \epsilon < b$ for all $n \geq N$, we have $\frac{1}{n} \in (a, b)$ for all $n \geq N$. Thus $\frac{1}{n} \to 0$.

Example. $\{n\}$ does not converge to any real number since any open interval (a, b) cannot contain a positive integer $> \lfloor b \rfloor$.

Theorem 25. (Cantor's intersection) *Let $I_1 \supseteq I_2 \supseteq \cdots I_n \supseteq I_{n+1} \cdots$ be a sequence of closed and bounded intervals in $\mathbb{R}$. Then $\cap_{n \in \mathbb{Z}_+} I_n \neq \emptyset$.*

Proof. Let for each $n \in \mathbb{Z}_+$, $I_n = [a_n, b_n]$ for some $a_n, b_n \in \mathbb{R}$. Clearly, the sequence $\{a_m\}$ is monotonically increasing. For any $m,\ n \in \mathbb{Z}_+$, $a_m \leq a_{m+n} < b_{m+n} \leq b_n$. Therefore each b_n is an upper bound of the set $A = \{a_m \mid m \in \mathbb{Z}_+\}$ in $\mathbb{R}$. By least upper bound property of $\mathbb{R}$, let $\sup A = \alpha \in \mathbb{R}$. Then $a_m \leq \alpha \leq b_n$ for all $m,\ n \in \mathbb{Z}_+$, i.e., $\alpha \in \cap_{n \in \mathbb{Z}_+} I_n$. ∎

Definition. (open and closed sets) A subset U of $\mathbb{R}$ is said to be *open* in $\mathbb{R}$, if for every $p \in U$, there is an open interval (a, b) containing p, such that $(a, b) \subseteq U$. A subset F of $\mathbb{R}$ is called *closed* if $\mathbb{R} - F$ is open. Let A be a subset of $\mathbb{R}$. A point $q \in \mathbb{R}$ is said to be a *limit point* of a set A, if for every open set $U \ni q$ in $\mathbb{R}$, $A \cap (U - \{q\}) \neq \emptyset$.

Example. Every open interval $(a, b),\ a,\ b \in \mathbb{R}$ is an open subset of $\mathbb{R}$.

Theorem 26. *Following three results are true regarding open subsets of $\mathbb{R}$.*
(a) $\emptyset$ and $\mathbb{R}$ are open in $\mathbb{R}$
(b) If $\{U_\alpha\}_{\alpha \in J}$ is a collection of open subsets of $\mathbb{R}$ then $\cup_{\alpha \in J} U_\alpha$ is open in $\mathbb{R}$.
(c) If the sets U and V are open in $\mathbb{R}$ then so is $U \cap V$.

Proof. (a) Since there is no point in $\emptyset$, it is open in $\mathbb{R}$. For any $x \in \mathbb{R}$, $x \in (x-1, x+1) \subsetneq \mathbb{R}$; so, $\mathbb{R}$ is open.
(b) Let $U = \cup_{\alpha \in J} U_\alpha$, and let $x \in U$. Then $x \in U_{\alpha_0}$ for some $\alpha = \alpha_0$. Since U_{α_0} is an open set containing x, there is an open interval $(a, b) \ni x$, such that $(a, b) \subseteq U_{\alpha_0} \subseteq U$. Therefore, U is open.
(c) Let $x \in U \cap V$. Since U and V are open sets containing the point x, there exist open intervals $(a, b) \ni x$ and $(c, d) \ni x$ such that $(a, b) \subseteq U$ and $(c, d) \subseteq V$. Let $\alpha = \max\{a, c\}$, and let $\beta = \min\{b, d\}$. Then $x \in (\alpha, \beta) \subseteq (U \cap V)$. Hence $U \cap V$ is open. ∎

Example. Let $a,\ b \in \mathbb{R}$. The set $(a, \infty) := \{x \in \mathbb{R} \mid a < x\}$ is open, since for any $x \in (a, \infty)$, $x \in (a, x+1) \subsetneq (a, \infty)$. Similarly, the set $(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$ is also open. The closed interval $[a, b]$ is a closed subset of $\mathbb{R}$, since $\mathbb{R} - [a, b] = (-\infty, b) \cap (a, \infty)$ is open by Theorem 26.

Example. The set $\mathbb{Z}$ is not open in $\mathbb{R}$ since $\mathbb{Z}$ does not contain any interval. Consider $\mathbb{R} - \mathbb{Z} = \cup_{n\in\mathbb{Z}}(n, n+1)$, which is open by Theorem 26. So, $\mathbb{Z}$ is closed subset of $\mathbb{R}$.

Theorem 27. (Bolzano-Weierstrass) *Every infinite set of real numbers, which is bounded above in $\mathbb{R}$, has a limit point in $\mathbb{R}$.*

Proof. Let A be an infinite subset of $\mathbb{R}$, which is bounded above in $\mathbb{R}$. By least upper bound property of $\mathbb{R}$, let $\sup A = \beta \in \mathbb{R}$. We will show that β is a limit point of A. So, let U be an open set containing β. Choose an open interval $(a, b) \ni \beta$, such that $(a, b) \subseteq U$. If $A \cap (a, \beta) = \emptyset$ then every $\alpha \in (a, \beta)$ becomes an upper bound of A, such that $\alpha < \beta$, which is a contradiction. Thus, $A \cap ((a, b) - \{\beta\}) \neq \emptyset$, and β is a limit point of A. ∎

Definition. (Compactness) A subset A of $\mathbb{R}$ is said to be compact if for every collection $\mathcal{C} = \{U_\alpha\}_{\alpha\in J\subseteq\mathbb{R}}$ of open subsets of $\mathbb{R}$, such that $A \subseteq \cup_{\alpha\in J} U_\alpha$, there is a finite sub-collection $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ satisfying $A \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. The collection $\mathcal{C}$ is called an open cover or open covering of A by sets open in $\mathbb{R}$.

Theorem 28. (Hein-Borel) *The interval $[a, b] \subseteq \mathbb{R},\ a,\ b \in \mathbb{R}$ is compact in $\mathbb{R}$.*

Proof. Let $\mathcal{C}$ be an open covering of the interval $[a, b]$ by sets open in $\mathbb{R}$. Define the set

$$A := \{x \in \mathbb{R} \mid [a, x] \text{ is covered by finitely many members of } \mathcal{C}\}.$$

Observe that $A \neq \emptyset$ since $a \in A$. If $b < x$ for some $x \in A$ then the closed interval $[a, b] \subseteq [a, x]$ is covered by finitely many members of $\mathcal{C}$, and we are done. So, assume that $x \leq b$. Then the set A is bounded above by b in $\mathbb{R}$. By least upper bound property of $\mathbb{R}$, let $\sup A = \alpha \in \mathbb{R}$.
Claim I: $\alpha \in A$. For every $x \leq \alpha$, such that $a < x$, $x \in A$. So, choose an open set $U_\alpha \in \mathcal{C}$ containing α, such that $(x, \alpha] \subseteq U_\alpha$. Then the closed interval $[a, \alpha] = [a, x] \cup (x, \alpha]$ is covered by finitely many members of $\mathcal{C}$. Thus, $\alpha \in A$.
Claim II: $\alpha = b$. If possible, let $\alpha < b$. Choose $y \in U_\alpha$, such that $\alpha < y < b$. By a similar argument as in Claim I above, $y \in A$, which leads to a contradiction. Thus $\alpha = b \in A$, such that $[a, b]$ is covered by finitely many members of $\mathcal{C}$, which proves the theorem. ∎

We make a pause here and examine the proofs of Theorems 25-28, which use *least upper bound property* of the real-line. All the deductions have one more thing in common; they are expressible in terms of *open subsets of* $\mathbb{R}$, which satisfy the three axioms of Theorem 26.

Definition. (Usual topology of $\mathbb{R}$) If the usual definition of an open set in $\mathbb{R}$ is assumed (see Page 19) then the collection $\mathcal{U}$ of all open subsets of $\mathbb{R}$ is called *the usual topology* of $\mathbb{R}$.

Exercises

1. Prove that the set of positive integers is infinite and satisfies the least upper bound property.

2. Prove that a well ordered set satisfies the least upper bound property.

3. Given a nonempty set A with an order $\prec$, if every nonempty subset of A has a smallest element then the relation $\prec$ is called a *well ordering* and the set A is called a *well ordered set*. Using Zorn's Lemma, prove the well ordering theorem: *if A is a nonempty set then there exists an order relation on A that is a well ordering.*

4. Prove that $\mathbb{R}$ is Hausdörff, i.e., for any two distinct real numbers x and y, there exist disjoint open sets U and V containing x and y, respectively.

5. A sequence $\{x_n\}$ of real numbers is said to be a Cauchy-sequence if for every $\epsilon > 0$, there is a positive integer N, such that $|x_m - x_n| < \epsilon$, for all $m, n \geq N$. Prove that every Cauchy sequence of real numbers converges in $\mathbb{R}$.

6. A sequence is said to be monotonically increasing, if it is an increasing function of positive integers. Prove that a monotonically increasing sequence of real numbers which is bounded above is convergent.

7. A subsequence $\{y_n\}$ of a sequence $\{x_n\}$ is a map obtained from $\{x_n\}$ by restricting the domain to an infinite subset of $\mathbb{Z}_+$. Prove that a sequence of real numbers is convergent if and only if its every subsequence converges to the same point.

8. Prove that a sequence of real numbers converges to at most one point.

9. Prove that the Cantor's intersection theorem and Bolzano-Weierstrass theorem are equivalent.

10. Prove the Hein-Borel theorem using Cantor's intersection theorem.

11. Consider the usual topology of $\mathbb{R}$. Prove that $\mathbb{Z}$ is a closed subset of $\mathbb{R}$. Is $\mathbb{Q}$ closed subset of $\mathbb{R}$? is $\mathbb{Q}$ open in $\mathbb{R}$?

12. Prove that every finite subset of $\mathbb{R}$ is closed in the usual topology of $\mathbb{R}$.

13. Prove that a subset A of $\mathbb{R}$ is closed if and only if it contains all of its limit points.

14. Prove that the open interval $(0, 1)$ is not compact in $\mathbb{R}$.

15. Consider the subset $S = \{\sin n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$. Determine the set of all limit points of S in $\mathbb{R}$.