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'Calculus of several variables'
Lecture notes on 'Derivatives as Linear maps'

In these notes, we will define the derivatives of functions of several variables $f : \mathbb{R}^n \to \mathbb{R}^m,\ m, n \in \mathbb{Z}^+$ and discuss their implications.

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1 Derivative as a Linear map

Definition: (Derivative) Let U be an open subset of $\mathbb{R}^n$. A map $f : U \to \mathbb{R}^m$ is said to be differentiable at point $x \in U$ if there is a $m \times n$ matrix $\alpha \in \mathbb{R}^m \times \mathbb{R}^n$ where for each $h \in \mathbb{R}^n$ such that $x + h \in U$, the following holds

$$\lim_{h\to 0} \epsilon(h) = \lim_{h\to 0} \frac{\|f(x + h) - f(x) - \alpha \cdot h\|}{\|h\|} = 0.$$

If the above limit exists then the matrix α is called the derivative (or **Jacobian matrix**) of f at x and we write $f'(x) = \alpha$.

Definition: (Derivative) Let U be an open subset of $\mathbb{R}^n$. A map $f : U \to \mathbb{R}^m$ is said to be differentiable at point $x \in U$ if there is linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that for each $h \in \mathbb{R}^n$ for which $x + h \in U$, the following holds

$$\lim_{h\to 0} \epsilon(h) = \lim_{h\to 0} \frac{\|f(x + h) - f(x) - \lambda(h)\|}{\|h\|} = 0.$$

If the above limit exists, then the linear map λ is called **the derivative of** f at x and we write $Df(x) = \lambda$.

Remark: Note that the two definitions above are equivalent in the sense that in the standard ordered basis of $\mathbb{R}^n$ there is a one to one correspondence between the set of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and the set of all $m \times n$ matrices over $\mathbb{R}$. More precisely, given a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$, there is a unique matrix α over $\mathbb{R}$ such that

$$L(x) = \alpha \cdot x, \text{ for all } x \in \mathbb{R}^n.$$

In this view, we have treated every element x of $\mathbb{R}^n$ as a $n \times 1$ matrix (column vector) and we will do the same everywhere here. Conversely, for any $m \times n$ matrix α over $\mathbb{R}$ there is unique linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(e_i) = \alpha \cdot e_i, \text{ for all } i = 1, ..., n$$

i.e. the matrix of T with respect to the standard basis of $\mathbb{R}^n$ is α.

Remark: If $f : U \to \mathbb{R}^m$ is differentiable at x, then it is continuous at x. To see this, let $Df(x) = \lambda$, then λ being a linear map, is a continuous function on $\mathbb{R}^n$, therefore it is continuous at x.

Notation: The symbol $f'(x)$, for $f : U \to \mathbb{R}^m$ is reserved for representing the Jacobian matrix of f at $x \in U$ and, the derivative of f at x will be denoted by $Df(x)$ such that the two are related by

$$Df(x)(h) = f'(x) \cdot h \text{ for all } h \in \mathbb{R}^n$$

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be such that $f(x, y) = (x^2, y^2, x + y)$. Then f is differentiable at any point (x, y) of $\mathbb{R}^2$. To see this, we need to search out a linear map as derivative of f. Let us try

$$\lambda(h) = \begin{pmatrix} 2x & 0 \\ 0 & 2y \\ 1 & 1 \end{pmatrix} \cdot h$$

Let $h = (h_1, h_2)^T$ then

$$\frac{\|f(x + h) - f(x) - \lambda(h)\|}{\|h\|}$$
$$= \frac{\|((x + h_1)^2 - x^2, (y + h_2)^2 - y^2, h_1 + h_2) - \lambda(h)\|}{\|h\|}$$
$$= \frac{\|(h_1^2, h_2^2, 0)\|}{\|h\|} \le \frac{\sqrt{(h_1^2 + h_2^2)^2}}{\|h\|} = \|h\| \to 0 \text{ as } h \to 0.$$

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This verifies that the linear map as defined above is indeed derivative of f at x. Here, the jacobian matrix is

$$f'(x) = \begin{pmatrix} 2x & 0 \\ 0 & 2y \\ 1 & 1 \end{pmatrix}.$$

Well, how to determine such a linear map as a derivative of f at x? We will do it soon, but first, let us establish uniqueness of the derivative.

Theorem 1.1. *Let $f : U \to \mathbb{R}^m$ be differentiable. Then the derivative $Df(x)$ at $x \in U$ is unique.*

Proof. Let λ and μ be two derivatives of f at x. Then they satisfy for each $h \in \mathbb{R}^n$ such that $x + h \in U$, the following

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - \lambda(h)\|}{\|h\|} = 0,$$

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - \mu(h)\|}{\|h\|} = 0.$$

Consider

$$\begin{aligned}\lim_{h\to 0} \frac{\|\lambda(h) - \mu(h)\|}{\|h\|} &\leq \lim_{h\to 0} \frac{\|f(x+h) - f(x) - \lambda(h)\|}{\|h\|} \\ &+ \lim_{h\to 0} \frac{\|f(x+h) - f(x) - \mu(h)\|}{\|h\|} = 0 \\ &= 0 + 0 = 0.\end{aligned}$$

Let $y \in U$. If $y = 0$ then $\lambda(0) = \mu(0) = 0$. If $y \neq 0$ then for $t \in \mathbb{R} - \{0\}$, using (1) we get

$$0 = \lim_{t\to 0} \frac{\|\lambda(ty) - \mu(ty)\|}{\|ty\|} = \lim_{t\to 0} \frac{\|\lambda(y) - \mu(y)\|}{\|y\|}$$

□

which proves that $\lambda(y) = \mu(y)$ for all $y \in U$.

Theorem 1.2. (Chain rule) *Let U be open subset of $\mathbb{R}^n$ and V be an open subset of $\mathbb{R}^m$ such that $f : U \to V$ is differentiable at x and that $g : V \to \mathbb{R}^p$ be differentiable at $f(x)$. Then the composition map $g \circ f : U \to \mathbb{R}^p$ is also differentiable at x; moreover,*

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x).$$

The Jacobian matrix for $D(g \circ f)(x)$ is given by

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

Proof. Let $Df(x) = \lambda,\ Dg(f(x)) = \mu$ and consider

$$\begin{aligned}\epsilon(h) &:= \frac{\|(g \circ f)(x+h) - (g \circ f)(x) - (\mu \circ \lambda)(h)\|}{\|h\|} \\ &\leq \frac{\|(g(f(x+h)) - g(f(x)) - \mu(f(x+h) - f(x))\|}{\|h\|} \\ &+ \left\| \mu\left(\frac{f(x+h) - f(x) - \lambda(h)}{\|h\|} \right) \right\| \\ &\leq \frac{\|(g(f(x+h)) - g(f(x)) - \mu(f(x+h) - f(x))\|}{\|h\|} \\ &+ M \frac{\|f(x+h) - f(x) - \lambda(h)\|}{\|h\|} \\ &\leq \frac{\|(g(f(x+h)) - g(f(x)) - \mu(f(x+h) - f(x))\|}{\|f(x+h) - f(x)\|} \\ &\times \left(\frac{\|f(x+h) - f(x) - \lambda(h)\|}{\|h\|} + \frac{\|\lambda(h)\|}{\|h\|} \right) \\ &+ M \frac{\|f(x+h) - f(x) - \lambda(h)\|}{\|h\|} \\ &\to 0 \text{ as } h \to 0\end{aligned}$$

where along with the linearity of μ, we have used the continuity of the function f at point x and the fact that every linear map on $T : \mathbb{R}^n \to \mathbb{R}^m$ is bounded i.e. there is a positive real number M such that $\|T(h)\| \leq M\|h\|$. Finally we have used the differentiability of g at $f(x)$ and differentiability of f at x. □

Theorem 1.3. *Let $f : U \to \mathbb{R}^m$ such that $f(x) = (f_1(x), ..., f_m(x))^T$, where $f_i : U \to \mathbb{R}$ are scalar functions. Then f is differentiable if and only if each f_i is differentiable at x. More over*

$$Df(x) = (Df_1(x), ..., Df_m(x))^T,$$

$$f'(x) = (f_1'(x), ..., f_m'(x))^T$$

where T denote matrix transpose. Thus $f'(x)$ is a $m \times n$ matrix whose rows are $f_i'(x)$ and therefore $(f'(x))_{i,j} = \frac{\partial f_i}{\partial x_j} = D_j f_i(x)$.

Proof. First note that $f_i = \pi_i \circ f$. Since π_i is a linear map, it follows that π_i is differentiable with π_i itself as the derivative. So, if f is differentiable at x then so is π_i at $f(x)$, and by the chain rule, we see that $\pi_i \circ f$ is differentiable at x.
Conversely, let each f_i is differentiable. Take

$$\lambda := (Df_1(x), ..., Df_m(x))^T$$

and consider

$$\epsilon(h) := \frac{\|f(x+h) - f(x) - \lambda(h)\|}{\|h\|}$$
$$\leq \sum_{i=1}^{m} \frac{\|f_i(x+h) - f_i(x) - Df_i(x)(h)\|}{\|h\|}$$
$$\to 0 \text{ as } h \to 0$$

since by differentiability of f_i for each $i = 1, ..., m$, at x

$$\frac{\|f_i(x+h) - f_i(x) - Df_i(x)(h)\|}{\|h\|} \to 0 \text{ as } h \to 0.$$

More over as $\lambda = Df(x) = (Df_1(x), ..., Df_m(x))^T$ therefore

$$\begin{aligned}\lambda(h) = f'(x) \cdot h &= (Df_1(x), ..., Df_m(x))^T(h) \\ &= (Df_1(x)(h), ..., Df_m(x)(h))^T \\ &= (f_1'(x), ..., f_m'(x))^T \cdot h \end{aligned} \tag{1.1}$$

as h is arbitrary, it follows that

$$f'(x) = (f_1'(x), ..., f_m'(x))^T$$

where we have each

$$f_i'(x) = \nabla f_i(x) = (D_1 f_i(x), ..., D_n f_i(x)))$$

which is a $1 \times n$ matrix as we have seen from the derivative of scalar functions. Thus $(f'(x))_{ij} = D_j f_i(x)$ as required. □

Remark: The preceding theorem provides us the description of the derivative of a vector valued function of several variables, namely, via the partial derivatives of the components of the map. Thus if $f : \mathbb{R}^3 \to \mathbb{R}^2$ is differentiable at (x, y, z) such that $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$ then

$$f'(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} (x, y, z)$$

which uniquely determines the linear transformation i.e. the derivative of f at x, whose matrix in the standard ordered basis is $f'(x)$.

Definition: A map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is said to be a bilinear map if for all $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in \mathbb{R}^m$, and $\alpha \in \mathbb{R}$ the following hold

1. $f(\alpha x, y) = \alpha f(x, y) = f(x, \alpha y)$
2. $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$
3. $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$

Proposition 1.4. *1. Let $f : U \to \mathbb{R}^m$ is constant, then $Df(x) = 0$ for all $x \in U$.*

2. Let $f : U \to \mathbb{R}^m$ is a linear map, then $Df(x) = f$ for all $x \in U$.

3. If $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a bilinear map then $Df(x, y)(h_1, h_2) = f(x, h_2) + f(h_1, y)$.

4. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ s.t. $f(x, y) = x^T \cdot y$ then $Df(x, y)(h_1, h_2) = x^T \cdot h_2 + h_1^T \cdot y$.

Proof. *1.* As f is constant $f(x+h) - f(x) = 0$ for all $h \in \mathbb{R}^n$. The zero map is a linear map and here $\epsilon(h) = \frac{\|f(x+h) - f(x) - 0(h)\|}{\|h\|} = 0$ This proves that $Df(x) = 0$.
2. Given that f is a linear map, therefore

$$\epsilon(h) = \frac{\|f(x+h) - f(x) - f(h)\|}{\|h\|} = \frac{\|f(0)\|}{\|h\|} = 0.$$

This proves $Df(x) = f$.
3. Consider for $(h_1, h_2) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $(x + h_1, y + h_2) \in U$,

$$\begin{aligned}\epsilon(h_1, h_2) &= \frac{\|f(x + h_1, y + h_2) - f(x, y) - f(x, h_2) - f(h_1, y)\|}{\|(h_1, h_2)\|} \\ &= \frac{\|f(x, h_2) + f(h_1, h_2) + f(h_1, y) - f(x, h_2) - f(h_1, y)\|}{\|(h_1, h_2)\|} \\ &= \frac{\|f(h_1, h_2)\|}{\|(h_1, h_2)\|} \to 0 \text{ as } h_1 \to 0 \text{ for all } h_2\end{aligned}$$

Similarly $\epsilon(h_1, 0) \to 0$ for all h_2. It follows that $\epsilon(h_1, h_2) \to 0$ as $(h_1, h_2) \to 0$.
4. Follows from *3.* □

Theorem 1.5. *Let $f : U \to \mathbb{R}^m$. Then $Df(x)$ exists if all partial derivatives $D_j f_i(x)$ exist and are continuous at $x \in U$.*

Proof. By theorem 1.3, $Df(x)$ exists if and only if $Df_i(x)$ exists for each i. Now $f_i : U \to \mathbb{R}$ is scalar function, therefore, $Df_i(x)$ exists if and only if each of the partial derivatives $D_j f_i(x)$ exist and are continuous at x. This completes the proof. □

Theorem 1.6. (Another form of chain rule) *Let $g_i : U \to \mathbb{R},\ i = 1, ..., m$, be continuously differentiable functions at $x \in U$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable function at $(g_1(x), ..., g_m(x))$. Then the function $F(x) = f(g_1(x), ..., g_m(x))$ is differentiable at x and that*

$$D_i F(x) = \sum_{j=1}^{m} D_j f(g_1(x), ..., g_m(x)) \cdot D_i g_j(x).$$

Proof. Let $g : U \to \mathbb{R}^m$ be $g(x) = (g_1(x), ..., g_m(x))^T$. Since each g_i is continuously differentiable, it follows that g is differentiable at x. Observe that $F = f \circ g$, therefore by the chain rule, we have F is differentiable at x and that

$$F'(x) = f'(g(x)) \cdot g'(x)$$

which gives

$$\begin{aligned} D_i F(x) &= F'(x) \cdot e_i = f'(g(x)) \cdot (g'(x) \cdot e_i) \\ &= (D_1 f(g(x)), ..., D_m f(g(x))) \cdot (D_i g_1(x), ..., D_i g_m(x))^T \\ &= \sum_{j=1}^{m} D_j f(g(x)) \cdot D_i g(x) \end{aligned}$$

which proves the result. □

1.1 Higher order derivatives

Definition: Let $M(m, n; \mathbb{R})$ denote the set of all $m \times n$ matrices over $\mathbb{R}$. We identify this space by $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{mn}$ in the standard topology.

Definition: Let $f : U \to \mathbb{R}^m$ be differentiable at "every" point $x \in U$. This defines a map $\mathcal{D}f : U \to M(m, n; \mathbb{R}) = \mathbb{R}^{mn}$ defined by $x \mapsto f'(x)$. $\mathcal{D}f$ is called the total derivative of f. If $\mathcal{D}f$ is differentiable at $x \in U$ then we say that f is twice differentiable at x. If f is twice differentiable at each $x \in U$, then it defines another map $\mathcal{D}^2 f : U \to M(mn, n; \mathbb{R}^n) = \mathbb{R}^{mn} \times \mathbb{R}^n$. More generally, if f is $(k-1)-$times differentiable and if $\mathcal{D}^{k-1} f : U \to \mathbb{R}^{mn^{k-1}} \times \mathbb{R}^n$ is differentiable, we say that f is $k-$times differentiable.

Example: If $f : \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))^T$. Then $\mathcal{D}f : \mathbb{R}^3 \to M(2, 3; \mathbb{R}) = \mathbb{R}^6$ is given by

$$\mathcal{D}f(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} (x, y, z)$$

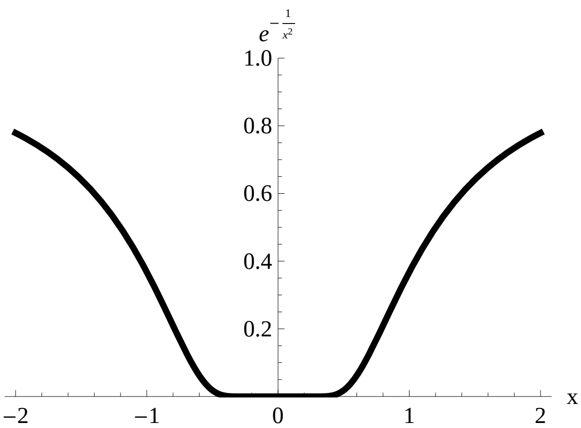
If $\mathcal{D}f$ is differentiable function of (x, y, z), then, $\mathcal{D}^2 f : \mathbb{R}^3 \to M(2 \times 3, 3; \mathbb{R}) = \mathbb{R}^6 \times \mathbb{R}^3$. We have

$$\mathcal{D}^2 f(x, y, z) = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x^2} & \frac{\partial^2 f_1}{\partial y \partial x} & \frac{\partial^2 f_1}{\partial z \partial z} \\ \frac{\partial^2 f_1}{\partial x \partial y} & \frac{\partial^2 f_1}{\partial y^2} & \frac{\partial^2 f_1}{\partial z \partial y} \\ \frac{\partial^2 f_1}{\partial x \partial z} & \frac{\partial^2 f_1}{\partial y \partial z} & \frac{\partial^2 f_1}{\partial z^2} \\ \frac{\partial^2 f_2}{\partial x^2} & \frac{\partial^2 f_2}{\partial y \partial x} & \frac{\partial^2 f_2}{\partial z \partial z} \\ \frac{\partial^2 f_2}{\partial x \partial y} & \frac{\partial^2 f_2}{\partial y^2} & \frac{\partial^2 f_2}{\partial z \partial y} \\ \frac{\partial^2 f_2}{\partial x \partial z} & \frac{\partial^2 f_2}{\partial y \partial z} & \frac{\partial^2 f_2}{\partial z^2} \end{pmatrix} (x, y, z)$$

1.1.1 Existence of nonegative C^∞ scalar functions with compact support

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{1.2}$$



The graph of the function in (1.2) is plotted in the figure that follows. Observe that the graph is smooth and, is flat near the point $x = 0$. It indicates that $f'(0) = 0$. Infact, we will show that $\frac{d^n f(0)}{dx^n} = 0$ for each positive integer n. We have $f'(0) = \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{x \to 0} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{x \to 0} \frac{x}{2e^{\frac{1}{x^2}}} = 0$ where we have used the L'Hospital's rule. To check continuity of f' at 0, we have $f'(x) = \frac{2}{x^2}e^{-\frac{1}{x^2}} \to 0$ as $x \to 0$ and is continuous at $x = 0$. $f'(x)$ is already continuous for all $x \neq 0$ as so is the composition of the exponential function and the function $x \mapsto \frac{-1}{x^2}$. We see that $f'(x)$ is continuous on $\mathbb{R}$. Now $f''(x) = 4\left(-\frac{1}{x^3} + \frac{1}{x^5}\right)e^{-\frac{1}{x^2}}$ which is continuous for all $x \neq 0$. At $x = 0$, $\lim_{x\to 0} f''(x) = 0$ as before and $f''(0) = \lim_{t\to 0} \frac{\frac{2}{x^2}e^{-\frac{1}{x^2}}}{x} = 0$ again by L'Hospital rule. While evaluating $n-$th derivative of $f(x)$ at $x \neq 0$ we have for all $n = 1, 2, ...$

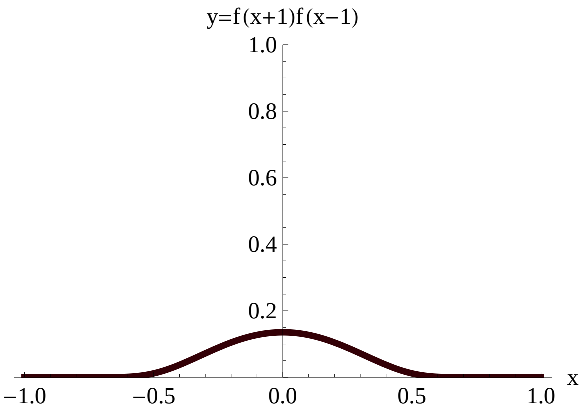
$$\frac{d^n f(x)}{dx^n} = \left(\frac{\alpha_1}{x^{n+1}} + \frac{\alpha_2}{x^{n+3}} + ... + \frac{\alpha_n}{x^{n+1+2(n-1)}}\right)e^{-\frac{1}{x^2}} \to 0 \text{ as } x \to 0$$

for some scalars $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and by induction, it follows that

$$\frac{d^n f(0)}{dx^n} := \lim_{x\to 0} \frac{\frac{d^{n-1}f(x)}{dx^{n-1}} - \frac{d^{n-1}f(0)}{dx^{n-1}}}{x} = 0.$$

This proves that f is $n-$times continuously differentiable for all $n = 1, 2, ...$, i.e. f is C^∞.
If we define $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) := \begin{cases} f(x+1)f(x-1) & \text{if } x \in (-1,1) \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$



Then g is C^∞ function which is positive on $(-1, 1)$ and zero on $\mathbb{R} - (0, 1)$. The only points which need investigation are $x = \pm 1$. To see this we evaluate $n-$th derivative of g which is given by

$$\frac{d^n g(x)}{dx^n} = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{d^r f(x+1)}{dx^r} \frac{d^{n-r} f(x-1)}{dx^{n-r}} \to 0$$

as $x \to \pm 1$ since f is continuously differentiable. Using induction we find that $\frac{d^n g(\pm 1)}{dx^n} = 0$. This establishes that g is C^∞. Since $f(x-1)f(x+1) = e^{-\frac{1}{(x+1)^2}}e^{-\frac{1}{(x-1)^2}} > 0$ on $(-1, 1)$ it follows that $g(x) > 0$ on $(-1, 1)$.

Proposition 1.7. *For any $\epsilon > 0$, there exists a nonnegative C^∞ function $h : \mathbb{R} \to [0, 1]$ such that $h(x) = 0$ for all $x \leq 0$, $h(x)$ is not constant on $(0, \epsilon)$, and $h(x) = 1$ for all $x \geq \epsilon$.*

Proof. We may define for $\epsilon > 0$, $g_\epsilon : \mathbb{R} \to \mathbb{R}$ such that

$$g_\epsilon(x) := \begin{cases} f(x)f(x-\epsilon) & \text{if } x \in (0,\epsilon) \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

so that g_ϵ is C^∞ function that is positive on the interval $(0, \epsilon)$ and zero elsewhere.
Now, define for any $\epsilon > 0$, $h_\epsilon : \mathbb{R} \to [0, 1]$ by

$$h_\epsilon(x) := \frac{\int_0^x g_\epsilon(t)dt}{\int_0^\epsilon g_\epsilon(t)dt} \tag{1.5}$$

which is the required function. □

Proposition 1.8. *Let $x \in \mathbb{R}^n$ and for $\epsilon > 0$, $G_\epsilon : \mathbb{R}^n \to \mathbb{R}$ be defined by*

$$G_\epsilon(t) = g\left(\frac{t_1 - x_1}{\epsilon}\right) \cdots g\left(\frac{t_n - x_n}{\epsilon}\right)$$

where g is given by (1.3). Then G_ϵ is a C^∞ function which is positive on the open rectangle $R := (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$.

Proof. Follows from (1.3) since each $g\left(\frac{t_i - x_i}{\epsilon}\right)$ is positive on the interval $\frac{t_i - x_i}{\epsilon} \in (-1, 1)$ or $t_i \in (x_i - \epsilon, x_i + \epsilon)$ and zero otherwise. □

Theorem 1.9. *If $U \subset \mathbb{R}^n$ is open and $C \subset U$ is compact, then there is a nonegative C^∞ function $G : U \to \mathbb{R}$ such that $G(x) > 0$ for $x \in C$ and $G = 0$ outside of some closed set contained in U.*

Proof. Take for $\epsilon > 0$ an open covering of C by the basis of open rectangles in $\mathbb{R}^n$, of the form $R(x, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$ for each $x \in C$, such that for a choice of ϵ, $\overline{R(x, \epsilon)} \subset U$ (this is always possible why?). By compactness of C, there are finitely many points $p_1, ..., p_k \in C$ for a positive integer k such that

$$C \subset R(p_1, \epsilon_1) \cup \cdots \cup R(p_k, \epsilon_k) \subset \overline{R(p_1, \epsilon_1)} \cup \cdots \cup \overline{R(p_k, \epsilon_k)} \subset U.$$

Let $V = R(p_1, \epsilon_1) \cup \cdots \cup R(p_k, \epsilon_k)$. Then

$$C \subset V \subset \bar{V} \subset U.$$

Define $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(x) = g\left(\frac{x_1 - \pi_1(p_i)}{\epsilon_i}\right) \cdots g\left(\frac{x_n - \pi_n(p_i)}{\epsilon_i}\right)$$

whenever $x \in R(p_i, \epsilon_i)$ where g is given by (1.3). Then by the preceding propostion, G is a C^∞ function which is positive on the open set $V \supset C$ and vanishes on $(\mathbb{R}^n - V) \supset (\mathbb{R}^n - \bar{V})$ as required. □

Corollary 1.10. *Let U be an open subset of $\mathbb{R}^n$. Then there exists a nonnegative smooth function $\varphi : U \to [0, 1]$ such that $\varphi(x) = 1$ on C and $\varphi = 0$ outside a closed set contained in U.*

Proof. From theorem 1.9, choose a C^∞ function $G : U \to \mathbb{R}$ such that $G(x) > 0$ for $x \in C$ and $G = 0$ outside some closed subset containing C and contained in U. Since G is positive on C, there is an $\epsilon > 0$ such that $G(x) \geq \epsilon$ for all $x \in U$. Then define

$$\varphi(x) = (h_\epsilon \circ G)(x)$$

which is the required function. Note that for all $x \in C$ $G(x) \geq \epsilon$ and therefore $h_\epsilon(G(x)) = 1$. And outside a closed set $\bar{V}$ as in the theorem 1.9, $G = 0$ and that $h_\epsilon(G(x)) = h(0) = 0$ as required. □

Remark: The corollary 1.10 will be used in construction of partitions of unity, while pursuing the studies on integration. You should memorise this result for its later use.

1.2 References:

1. M. Spivak. *Calculus on Manifolds*, Addison Wesley (1965)

2. A. R. Shastri. *Elements of Differential Topology*, CRC Press (Taylor and Francis) (2011).

Exercises

1. Prove that for any positive integer n,

$$\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0.$$

2. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be defined by $f(x, y) = \langle x, y \rangle$. Determine $Df(x, y)$ and $f'(x, y)$.

3. If $f, g : \mathbb{R} \to \mathbb{R}^n$ are differentiable and $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(t) = \langle f(t), g(t) \rangle,$$

show that

$$h'(t) = \left\langle f'(t)^T, g(t) \right\rangle + \left\langle f(t), g'(t)^T \right\rangle.$$

4. If $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable and $\|f(t)\| = 1$ for all t, show that $\left\langle f'(t)^T, f(t) \right\rangle = 0$.

5. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) := \int_0^x g_1(t, 0)dt + \int_0^y g_2(x, t)dt.$$

   (a) Show that $D_2 f(x, y) = g_2(x, y)$

   (b) How should f be defined so that $D_1 f(x, y) = g_1(x, y)$?

   (c) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D_1 f(x, y) = x$ and $D_2 f(x, y) = y$.

   (d) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D_1 f(x, y) = y$ and $D_2 f(x, y) = x$.

6. If $f : \mathbb{R}^2 \to \mathbb{R}$ and $D_2 f = 0$, show that f is independent of the second variable. If $D_1 f = D_2 f = 0$, show that f is constant.

7. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x < 0, \text{ or } x \geq 0 \text{ and } y \neq 0\}$.

   (a) If $f : U \to \mathbb{R}$ and $D_1 f = D_2 f = 0$, show that f is constant.

   (b) Determine a function $f : U \to \mathbb{R}$ such that $D_2 f = 0$ but f is not independent of the second variable.

8. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

   (a) Show that $D_2 f(x, 0) = x$ for all x and $D_1 f(0, y) = -y$ for all y.

   (b) Show that $D_{1,2} f(0, 0) \neq D_{2,1} f(0, 0)$