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'Calculus of several variables'
Lecture notes on 'Differentiation'

In these notes, we will discuss the derivatives of a scalar function of several variables defiend on an open subset of $\mathbb{R}^n$.

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1 Directional derivative

Let U be an open subset of $\mathbb{R}^n$ in the product topology. This means for every point $x \in U$, there is an open rectangle R containing x such that $R \subseteq U$.

Definition: Any $f : U \to \mathbb{R}$ is called a scalar function.

Definition: (Directional derivative) Let $x \in U$ and $u \in \mathbb{R}^n$ such that $\|u\| = 1$, then the straight line passing through x, in the direction of u is the set $\{x+tu \mid t \in \mathbb{R}\}$ which intersects U. Let $I = \{t \in \mathbb{R} \mid x + tu \in U\}$, then $0 \in I$. Let $g : I \to \mathbb{R}$ be such that $g(t) := f(x+tu)$ and g is differentiable at $t = 0$. Then the directional derivative of f at point x in the direction of u is defined as

$$D_u f(x) = g'(0) = \lim_{t\to 0} \frac{f(x + tu) - f(x)}{t}.$$

Definition: (Partial derivative) Let $\{e_1, ..., e_n\}$ be the standard ordered basis of the vector space $\mathbb{R}^n(\mathbb{R})$. If $D_{e_i} f(x)$ exists for some $i = 1, 2..., n$, then it is called the partial derivative of f with respect to the $i-$th coordinate $x_i = \pi_i(x)$ at point x. We write $D_{e_i} f(x) = \frac{\partial f}{\partial x_i}$.

Example: Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x, y) := \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We will show that the directional derivative $D_u f(0, 0)$ exists for all $u \in \mathbb{R}^2$ such that $\|u\| = 1$; however, the function g is not continuous at $(0, 0)$. To see the discontinuity, observe that on any open set $U \ni (0, 0)$ if we choose the curve $\gamma(t) = (t, mt^2),\ m > 0$ then the graph $G_\gamma := \{\gamma(t) \mid t \in \mathbb{R}\}$ of the curve intersects U. On each point $(x, y) \in U \cap G_\gamma$,

$$\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{x\to 0} \frac{m^2}{1 + m^2} = \frac{m^2}{1 + m^2} \neq g(0, 0) = 0.$$

Infact the $lim_{p\to 0} g(p)$ does not exist since it is not unique.

On the other hand the directional derivative of g in the direction of $u = (u_1, u_2)$ exists at $(0, 0)$ and is equal to

$$\begin{aligned} D_u g(0, 0) &= \lim_{t\to 0} \frac{g((0, 0) + t(u_1, u_2)) - g(0, 0)}{t} \\ &= \lim_{t\to 0} \frac{t^2 u_1^2 u_2}{t^4 u_1^4 + t^2 u_2^2} \\ &= \begin{cases} \dfrac{u_1^2}{u_2} & \text{if } u_2 \neq 0, \\ 0 & \text{if } u_2 = 0 \end{cases} \end{aligned} \tag{1.1}$$

The preceding example shows that even if the directional deriavtive of a scalar function of several variables exists at a point, it does not guarantee that the fucntion will be continuous at that point! So the directional derivative can not be the general definition of the derivative of scalar functions. We are looking for a definition of derivative in which a differentiable function should be continuos! So we revise the differentiablity in functions of one variables and try to generalize it to the functions of sevaral variables.

Theorem 1.1. (Increment theorem(one variable)) *A function $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x \in (a, b)$ if and only if for each $h \in \mathbb{R}$ such that $x + h \in (a, b)$, there is an error function $\epsilon(h)$ and a real number $\alpha = f'(x) \in \mathbb{R}$ such that*

$$f(x + h) = f(x) + \alpha h + h\epsilon(h)$$

where $\epsilon(h) \to 0$ whenever $h \to 0$.

Proof. Define $\epsilon(h) := \begin{cases} \dfrac{f(x + h) - f(x)}{h} - \alpha & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$, then note that $\lim_{h\to 0} \epsilon(h) = 0$ if and only if $\lim_{h\to 0} \dfrac{f(x + h) - f(x)}{h} - \alpha = 0$. □

The proof of the last theorem motivates the next definition.

Definition: A function $f : U \to \mathbb{R}$ is said to be differentiable at a point $x \in U$ if there is an $\alpha \in \mathbb{R}^n$ such that for $x + h \in U$,

$$\lim_{h\to 0} \frac{f(x + h) - f(x) - \alpha \cdot h}{\|h\|} = 0.$$

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If f is differentiable at $x \in U$ then α as above, is called derivative of f at x and, it is denoted by $Df(x)$.

Theorem 1.2. (Increment theorem(several variables)) *A map $f : U \to \mathbb{R}$ is differentiable at a point $x \in U$ if and only if for all $h \in \mathbb{R}^n$ such that $x + h \in U$, there is an error function $\epsilon(h)$ and an $\alpha \in \mathbb{R}^n$ satisfying*

$$f(x+h) = f(x) + Df(x) \cdot h + \|h\|\epsilon(h)$$

with $\epsilon(h) \to 0$ as $h \to 0$.

Proof. Proof is on the same steps as in the increment theorem and is left as an exercise. □

Theorem 1.3. *Let $f : U \to \mathbb{R}$ be differentiable at point x with derivative $Df(x)$. Then f is continuous at x.*

Proof. Using the theorem 1.2, we have an error function $\epsilon(h)$ such that for $x + h \in U$

$$f(x+h) = f(x) + Df(x) \cdot h + \|h\|\epsilon(h)$$

with $\epsilon(h) \to 0$ as $h \to 0$. Then

$$\lim_{h \to 0} f(x+h) = f(x) + Df(x) \cdot 0 + 0 \times 0 = f(x),$$

which proves continuity of f at x. □

Remark: Note that the derivative $Df(x) \in \mathbb{R}^n$; and that the directional derivative $D_{e_i}f(x)$ can be evaluated from the derivative of f just by taking $h = te_i$, $t \in \mathbb{R} - \{0\}$ such that as $h \to 0$ then $t \to 0$. It follows that

$$D_{e_i}f(x) = Df(x) \cdot e_i$$

which establishes a relation between the derivative $Df(x)$ and the partial derivatives $D_{e_i}f(x)$. Denoting $D_{e_i}f(x)$ by $D_if(x)$, $Df(x)$ is completely determined by the partial derivatives of f and we have

$$Df(x) = (D_1f(x), ..., D_nf(x)).$$

Now the question arises, how the continuity of f is related to the partial derivatives of f? We have the following result.

Notation: For $f : U \to \mathbb{R}$, the partial derivative $D_{e_i}f(x) = \frac{\partial f(x)}{\partial x_i}$ is denoted $D_if(x)$. Second order partial derivatives of f are denoted by $D_{1,2}f(x) = D_1(D_2f(x)) = \frac{\partial^2 f(x)}{\partial x_1 \partial x_2}$, $D_{1,1}f(x) = \frac{\partial^2 f(x)}{\partial x_1^2}$ etc. Similarly $D_{i_1 i_2 \dots i_k}f(x)$, $i_k, k \in \mathbb{Z}^+$, is a $k-$th order partial derivative of f at x and

$$D_{i_1 i_2 \dots i_k}f(x) = \frac{\partial^k f(x)}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$

Theorem 1.4. *Let all partial derivatives of $f : U \to \mathbb{R}$ exist and are bounded in U. Then f is continuous.*

Proof. It is given that for each $i = 1, 2, ..., n$, $|D_if(x)| < M$ for some positive real M. Consider for any $x, y \in U$,

$$\begin{aligned}
|f(x) - f(y)| &= |f(x_1, ..., x_n) - f(y_1, ..., y_n)| \\
&= |f(x_1, ..., x_n) - f(y_1, x_2, ..., x_n) \\
&\quad + f(y_1, x_2, ..., x_n) - f(y_1, y_2, x_3, ..., x_n) \\
&\quad \vdots \\
&\quad + f(y_1, ... y_{n-1}, x_n) - f(y_1, ..., y_n)| \\
&= |(x_1 - y_1)D_1f(c_1, x_2, ..., x_n) \\
&\quad + (x_2 - y_2)D_2f(y_1, c_2, ..., x_n) \\
&\quad \vdots \\
&\quad + (x_n - y_n)D_nf(y_1, y_2, ..., c_n)| \\
&\leq M \sum_{i=1}^{n} |x_i - y_i| \\
&\leq Mn|x - y| \to 0 \text{ as } x \to y
\end{aligned}$$

which proves continuity of f, where we have used the mean value theorem to choose c_i between x_i and y_i whenever $x_i \neq y_i$. □

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \sqrt{x^2 + y^2}\dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then the partial derivative $D_1f(0,0) = \lim_{t \to 0} \frac{f(t(1,0)) - f(0,0)}{t} = 0$ and $D_2f(0,0) = 0$. Also the function f is continuous at $(0,0)$. However $Df(0,0) = \lim_{(x,y) \to (0,0)} \frac{f(x,y) - f(0,0)}{\|(x,y)\|}$ does not exist! Thus even if all the partial derivative of f exist at a point of continuity of a function, the function may not be differentiable. *What extra condition is required to get differentiablity of a function via existence of its partial derivatives*? The next theorem answers this.

Definition: A map $f : U \to \mathbb{R}$ is said to be C^r, $r = 1, 2, ...$ if the partial derivatives of f of all orders less than or equal to r exist and are continuous. If f is C^r for every $r = 1, 2, ...$ then f is called a C^∞ function. A C^r function is also called smooth function.

Theorem 1.5. *If $f : U \to \mathbb{R}$ is such that all its partial derivatives exist in U and are continuous at $p \in U$, then f is differentiable at p. Moreover, the derivative $Df = (D_1f, ..., D_nf)$ itself is a continuous function on U.*

Proof. Take an open rectangle $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$ containing x and contained in U. Within this rectangle, consider the following for each $y \in R$, using the mean value theorem of one variable calculus and suitably choose c_i between x_i and y_i whenever $x_i \neq y_i$ such that $c_i \in (a_i, b_i)$ and the point of the form $(y_1, ...y_{i-1}, c_i, x_{i+i}, ..., x_n) \in R$.

$$\begin{aligned}
&|f(y) - f(x) - (D_1 f(x), ..., D_n f(x)) \cdot (y - x)| \\
&\quad \leq |(y_1 - x_1) D_1 f(c_1, y_2, ..., y_n) \\
&\quad + (y_2 - x_2) D_2 f(x_1, c_2, y_3, ..., y_n) \\
&\quad \vdots \\
&\quad + (y_n - x_n) D_n f(x_1, x_2, ..., c_n) - \sum_{i=1}^{n} D_i f(x)(y_i - x_i)| \\
&\quad \leq \|y - x\| \{|D_1 f(c_1, y_2, ..., y_n) - D_1 f(x)| \\
&\quad + |D_2 f(x_1, c_2, y_3, ..., y_n) - D_2 f(x)| \\
&\quad \vdots \\
&\quad + |D_n f(x_1, x_2, ...x_{n-1}, c_n) - D_n f(x)|\} \\
&\quad \to 0 \text{ as } y \to x
\end{aligned}$$

Since as $y \to x$, $c_i,\ y_i \to x_i$, thus

$$|D_1 f(c_1, x_2, ..., x_n) - D_1 f(x)| \to 0,$$
$$..., |D_n f(y_1, y_2, ..., c_n) - D_n f(x)| \to 0$$

because the partial derivatives are given to be continuous at x. We have proved that

$$\lim_{y-x \to 0} \frac{f(y) - f(x) - (D_1 f(x), ..., D_n f(x)) \cdot (y - x)}{\|y - x\|} = 0.$$

The proof of the theorem follows now. ☐

Proposition 1.6. *Let $f : U \to \mathbb{R}$ is differentiable at each point of U. Then at a point $x \in int(A)$ of local minimum or local maximum, each $D_i f(x) = 0$.*

Proof. Suppose f has a local extreme value at point $(x_1, ..., x_n)$. Define $g_i : \mathbb{R} \to \mathbb{R}$, s.t. $g_i(t) = f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_n)$. Then $g_i(x_i) = f(x_1, ..., x_n)$ which is the extereme value of f and hence of g_i. Since g_i is differentiable at each x_i where it attains its extrema, by one variable calculus, $0 = g_i'(x_i) = D_i f(x)$. ☐

Theorem 1.7. (Second order mean value theorem) *Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ such that $D_1 f, D_{2,1} f$ exist in U. Let $R = (a, a+h) \times (b, b+k)$ be the rectangle with vertices $(a, b), (a+h, b), (a+h, b+k)$, and $(a, b+k)$ such that $\bar{R} \subset U$. Then there is a point $(c_1, c_2) \in R$ such that*

$$hk D_{2,1} f(c_1, c_2) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b).$$

Here $D_{2,1} f = D_2(D_1 f)$.

Proof. Let $g : [a, a+h] \to \mathbb{R}$ s.t. $g(t) = f(t, b+k) - f(t, b)$. Then g is continuous on $[a, a+h]$ and differentiable function on $(a, a+h)$ since $D_1 f$ exists on U. Therefore, using mean value theorem of one variable calculus, there is a $c_1 \in (a, a+h)$ such that

$$\begin{aligned}
&hg'(c_1) = g(a+h) - g(a) \\
&\Rightarrow h(D_1 f(c_1, b+k) - D_1 f(c_1, b) \\
&\quad = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)
\end{aligned} \tag{1.2}$$

Now if we define $G : [b, b+k] \to \mathbb{R}$ by $G(s) = D_1 f(c_1, s)$, then $G'(s) = D_{2,1} f(c_1, s)$ which is given to exists as $(c_1, s) \in U$. Again by the mean value theorem, there is a $c_2 \in (b, b+k)$ such that

$$\begin{aligned}
&kG'(c_2) = G(b+k) - G(b) \\
&\Rightarrow k D_{2,1} f(c_1, c_2) = D_1 f(c_1, b+k) - D_1 f(c_1, b) \\
&\Rightarrow hk D_{2,1} f(c_1, c_2) = h(D_1 f(c_1, b+k) - D_1 f(c_1, b)) \\
&\Rightarrow hk D_{2,1} f(c_1, c_2) = f(a+h, b+k) - f(a+h, b) \\
&\qquad - f(a, b+k) + f(a, b) \quad \{\text{by}(1.2)\}
\end{aligned} \tag{1.3}$$

where the point $(c_1, c_2) \in R$. ☐

Corollary 1.8. *Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ such that $D_1 f,\ D_2, D_{2,1} f$ exist in U. If $D_{2,1} f$ is continuous at a point $p \in U$, then $D_{1,2} f(p)$ also exists and $D_{2,1} f(p) = D_{1,2} f(p)$.*

Proof. Let $\epsilon > 0$ be given. Using continuity of $D_{2,1}$, choose an open rectangle R with vertices (a, b), $(a+h, b)$, $(a+h, b+k)$, $(a, b+k)$, such that $p = (a, b) \in \bar{R} \subset U$ and

$$|D_{2,1}(p) - D_{2,1}(q)| < \epsilon \text{ for all } q \in R. \tag{1.4}$$

Also using the theorem 1.7, there exist $q = (c_1, c_2) \in R$ such that $\Delta f = hk D_{2,1} f(c_1, c_2) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$. Fix h and let $k \to 0$. Since $D_2 f$ exists in U, we have

$$\lim_{k \to 0} \frac{\Delta f}{hk} = \frac{D_2 f(a+h, b) - D_2 f(a, b)}{h} \tag{1.5}$$

☐

To prove existence of $D_{1,2} f(p)$, we just need to show that $\lim_{h \to 0}(\lim_{k \to 0} \frac{\Delta f}{hk})$ exists. For this consider

$$\begin{aligned}
&\left| \frac{D_2 f(a+h, b) - D_2 f(p)}{h} - D_{2,1} f(p) \right| \\
&\leq \left| \frac{D_2 f(a+h, b) - D_2 f(p)}{h} - \frac{\Delta f}{hk} \right| + \left| \frac{\Delta f}{hk} - D_{2,1} f(p) \right| \\
&= \left| \frac{D_2 f(a+h, b) - D_2 f(p)}{h} - \frac{\Delta f}{hk} \right| + \left| D_{2,1} f(q) - D_{2,1} f(p) \right| \\
&\to 0 \text{ as } h \to 0 (q \to p) \quad \{\text{by (1.5) and (1.4)}\}
\end{aligned}$$

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then check that $D_{1,2}f(0,0)$ and $D_{2,1}f(0,0)$ exist but are not equal. Explain why the mixed derivatives are not equal here?

An immediate generalization of the theorem 1.7 to $\mathbb{R}^n$ is the following

Theorem 1.9. (Higher order mean value theorem) *Let U be an open subset of $\mathbb{R}^n$ and R be an open rectangle with $\bar{R} \in U$ and vertices $v_1, ..., v_{2n}$ taken in order. If all order partial derivatives of $f : U \to \mathbb{R}$ exist in U upto n and are continuous at $v_1 \in U$, then there is a $c \in R$ such that*

$$\prod_{i=1}^{n} \pi_i(v_{i+1} - v_i) D_{n,n-1,...,1} f(c) = \sum_{i=1}^{n} (-1)^{i-1} f(v_{n-i})$$

where $\pi_i(x_1, ..., x_n) = x_i$ is the projection map.

Theorem 1.10. (Chain rule) *Let V be an open subset in $\mathbb{R}^n$. Let $f : (a,b) \to V$ and $g : V \to \mathbb{R}$ be such that $f = (f_1, ..., f_n)$ is differentiable at $t \in (a,b)$ and g is differentiable at $f(t)$. Then the composite map $g \circ f : (a,b) \to \mathbb{R}$ is also differentiable at t. More over*

$$(g \circ f)'(t) = Dg(f(t)) \cdot f'(t).$$

Proof. Using increment theorem we have for all $(t+s) \in (a,b)$ and $(f(t)+h) \in V$

$$f(t+s) - f(t) = f'(t)s + s\epsilon_1(s)$$

$$g(f(t)+h) - g(f(t)) = Dg(f(t)) \cdot h + \|h\| \epsilon_2(h)$$

with $\epsilon_1(s) \to$ as $s \to 0$ and $\epsilon_2(h) \to 0$ as $h \to 0$. If we set $h = h(s) := f(s+t) - f(t)$ then by conitnuity of f at t, $h(s) \to 0$ as $s \to 0$. Moreover, we have

$$\lim_{s \to 0} \frac{\|h(s)\|}{s} = \pm \|f'(t)\|$$

which means $\frac{h(s)}{s}$ is a bounded function. Observe that

$$\begin{aligned}
&(g \circ f)(t+s) - (g \circ f)(t) \\
&\quad = Dg(f(t)) \cdot h(s) + \|h(s)\| \epsilon_2(h(s)) \\
&\quad = Dg(f(t)) \cdot (f(t+s) - f(t)) + \|h(s)\| \epsilon_2(h(s)) \\
&\quad = Dg(f(t)) \cdot (f'(t) + s\epsilon_1(s)) + \|h(s)\| \epsilon_2(h(s)) \\
&\quad = Dg(f(t)) \cdot f'(t) + s\epsilon(s),
\end{aligned}$$

where

$$\epsilon(s) := Dg(f(t)) \cdot \epsilon_1(s) + \frac{\|h(s)\|}{s} \epsilon_2(h(s))$$

such that $\epsilon \to 0$ as $s \to 0$. This proves the result. □

Definition: A subset S of $\mathbb{R}^n$ is said to be convex if for each $x, y \in S$, $(tx + (1-t)y) \in S$ for all $t \in (0,1)$.

Theorem 1.11. (Taylor's theorem) *Let $f : U \to \mathbb{R}$ be a C^r function where U is a convex open subset (e.g. open rectangle, open ball etc.) of $\mathbb{R}^n$ containing a point x. Then for each $x + h \in U$, $h \in \mathbb{R}^n$, we have*

$$f(x+h) = f(x) + \sum_{k=1}^{r-1} \frac{1}{k!} \sum D_{i_1 i_2 \ldots i_k} f(x) h_{i_1} \cdots h_{i_k} + R(h) \tag{1.6}$$

where the remainder $R(h)$ satisfies $\lim_{h \to 0} \frac{R(h)}{\|h\|^{r-1}} = 0$ i.e. $R(h) = o(\|h\|^{r-1})$.

Proof. Fix $x \in U$ and define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = f(x+th).$$

Note that g is well defined since

$$x + th = (1-t)x + t(x+h) \in U$$

as $x, x+h \in U$ and U is convex. Using the Taylor theorem of one variable calculus, there is a $t_h \in (0,1)$ such that

$$g(1) = g(0) + \sum_{k=1}^{r-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(r)}(t_h)}{r!}. \tag{1.7}$$

Note that for each $k = 1, ..., r$, with a repeated application of the chain rule we get,

$$g^{(k)}(t) = \sum_{i_1, i_2, \ldots, i_k} D_{i_1 i_2 \ldots i_k} f(x+th) h_{i_1} h_{i_2} \cdots h_{i_k}$$

and, by taking $R(h) := \frac{g^{(r)}(t_h)}{r!}$ we get (1.8). Finally, consider

$$\begin{aligned}
r! \lim_{h \to 0} \frac{R(h)}{\|h\|^{r-1}} &= \lim_{h \to 0} \sum_{i_1, i_2, \ldots, i_r} D_{i_1 i_2 \ldots i_r} f(x + t_h h) \frac{h_{i_1} h_{i_2} \cdots h_{i_r}}{\|h\|^{r-1}} \\
&= \sum_{i_1, i_2, \ldots, i_r} D_{i_1 i_2 \ldots i_r} f(x) \lim_{h \to 0} \frac{h_{i_1} h_{i_2} \cdots h_{i_r}}{\|h\|^{r-1}} \\
&= 0
\end{aligned}$$

and this completes the proof. □

Corollary 1.12. (Lagrange's Mean value theorem) *Let $f : U \to \mathbb{R}$ be a C^1 function where U is a convex open subset (e.g. open rectangle, open ball etc.) of $\mathbb{R}^n$ containing a point x. Then for each $x + h \in U$ such that $h \in \mathbb{R}^n$, there is a $\theta \in (0,1)$ such that*

$$f(x+h) = f(x) + Df(x + \theta h) \cdot h \tag{1.8}$$

There is further generalization of the Taylor theorem, which we state in the next result.

Theorem 1.13. *Let $f : U \to \mathbb{R}$ be a C^r function on a convex open subset $U \subset \mathbb{R}^n$. Let $x \in U$, then there exist C^{r-1} functions $g_i : U \to \mathbb{R}$ satisfying $Df_i(x) = g(x)$ and that for each $h \in \mathbb{R}^n$ such that $x + h \in U$, we have*

$$f(x+h) = f(x) + \sum_{i=1}^{n} h_i g_i(x+h) \qquad (1.9)$$

Proof. Using fundamental theorem of one variable calculus, we have

$$f(x+h)-f(x) = \int_{[0,1]} \frac{df(x+th)}{dt} = \int_{[0,1]} Df(x+th)\cdot h dt$$

Now define

$$g_i(x+h) = \int_{[0,1]} D_i f(x+th)dt$$

which gives (1.9) where we see that

$$g_i(x) = D_i f(x) \int_{[0,1]} dt = D_i f(x).$$

This completes the proof. □

1.1 References:

1. M. Spivak. *Calculus on Manifolds*, Addison Wesley (1965)

2. A. R. Shastri. *Elements of Differential Topology*, CRC Press (Taylor and Francis) (2011).

Exercises

1. Obtain the directional derivative of $f : \mathbb{R}^3 - \{0\} \to \mathbb{R}$ s.t. $f(x,y,z) = (x^2+y^2+z^2)^{-1}$ at point $v = (1,1,1)$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) = -1$ and $\frac{\partial^2 f}{\partial y \partial x}(0,0) = 1$. Prove that $\frac{\partial^2 f}{\partial x \partial y}$ is not continuous at $(0,0)$.

3. Let $f, g : U \to \mathbb{R}$ be C^2 function and $x \in U$, where U is an open subset of $\mathbb{R}^n$. Prove the following

   (a) $D(fg) = gDf + fDg$

   (b) $D(f/g) = \frac{gDf - fDg}{g^2}, \; g(x) \neq 0$

   (c) $D^2(fg) = gD^2 f + 2DfDg + fD^2 g$

   Here $D = \nabla$ is called the gradient operator.

4. Divergence of $F : U \to \mathbb{R}^n$ at a point $x \in U$ is defined as $\nabla \cdot F = \sum_{i=1}^{n} D_i F \cdot e_i$. Prove the following formulas for smooth functions $f, g : U \to \mathbb{R}$.

   (a) $div(kF) = kdivF,\; k$ a constant

   (b) $div(fF) = fdiv(F) + \nabla f \cdot F$

   (c) $div(f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$

   (d) $div(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

5. For a smooth function $G : \mathbb{R}^3 \to \mathbb{R}^3$ such that $G = (g_1, g_2, g_3)$, define $curlG$ by

$$\begin{aligned} curlG = \nabla \times G :& \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)e_1 + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)e_2 \\ &+ \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)e_3 \end{aligned}$$

Show that $curl(\nabla f) = 0$ for a smooth scalar function $f : U \to \mathbb{R}$.

6. A smooth function $f : U \to \mathbb{R}$ is called homogeneous funcion of degree $\mu \in \mathbb{R}$ if $f(tx) = t^\mu f(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Prove the following Euler's identity for a homogeneous smooth function f of degree μ, $\sum_{i=1}^{n} x_i D_i f(x) = \mu f(x)$ where $x_i = \pi_i(x)$.

7. Let $f : U \to \mathbb{R}$ be a C^r, $r \geq 2$, function where U is a convex open subset of $\mathbb{R}^2$. Let $(x,y) \in U$ then show that the following holds

$$f(x+h, y+k) - f(x,y) = \sum_{k=1}^{r-1} \frac{1}{k!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^k f(x,y) + R(h,k)$$

for all $(x+h, y+k) \in U$ such that

$$R(h,k) = o(\|(h,k)\|^{r-1})$$

and

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^k f(x,y) = \sum_{i=1}^{k} \frac{k!}{i!(k-i)!}\frac{\partial^k f(x,y)}{\partial x^i y^{k-i}}.$$