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'Calculus of several variables'
Lecture notes & assignment #1

These are the preliminary notes on the basic vector space structure and the inner product space structure on $\mathbb{R}^n,\ n = 1, 2, ...$ These mathematical structures will allow us to do the calculus on functions of several variables. Throughout, we consider $\mathbb{R}^n$ as a vector space over $\mathbb{R}$ equipped with the component wise addition of vectors and component wise multiplication by scalar.

0.1 Inner product on $\mathbb{R}^n$

Definition: A point P in $\mathbb{R}^n$ is identified by a $n-$tuple $(x^1, ..., x^n)$ where each $x_i \in \mathbb{R},\ i = 1, ..., n$ is called the $i-$th coordinate of the point P. The point P therefore is identified with its position vector. The usual scalar product of the vectors in $\mathbb{R}^n$ defined by

$$\langle x, y\rangle = x \cdot y = x^1y^1 + \cdots + x^ny^n$$

is an inner product on $\mathbb{R}^n$ and will be called the standard inner product on $\mathbb{R}^n$. Norm of a vector x in $\mathbb{R}^n$ is

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

It immediately follows from the definition of the standard inner product that $\langle x, x\rangle \geq 0$ and $\langle x, y\rangle = 0$ if and only if $x = 0$.

Theorem 0.1. (Cauchy Schwarz inequality) *Let $(X, \langle\cdot\rangle)$ be an inner product space. Then for any $x, y \in X$,*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle .$$

Proof. Consider the polynomial

$$p(\lambda) := \langle x, x\rangle \lambda^2 + 2\langle x, y\rangle \lambda + \langle y, y\rangle$$

in λ which is a continuous and differentiable function of λ. Note that $p(\lambda) = \langle x + \lambda y, x + \lambda y\rangle \geq 0$ as seen in the earlier definition. Therefore any extreme value of $p(\lambda) \geq 0$ which occurs when $p'(\lambda) = 0$ i.e. for $\lambda = -\frac{\langle x,y\rangle}{\langle x,x\rangle}$. This means

$$p\left(-\frac{\langle x, y\rangle}{\langle x, x\rangle}\right) \geq 0$$

from which the inequality follows. □

Definition: For any two nonzero vectors $x, y \in \mathbb{R}^n$ we define angle theta between them to be

$$\theta = \cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right).$$

We say that x is perpendicular to y and write $x \perp y$ if $\langle x, y\rangle = 0$.

Definition: A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be:

1. norm preserving if $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$;
2. inner product preserving if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}$;
3. angle preserving if T is injective and $\cos^{-1}\left(\frac{\langle T(x), T(y)\rangle}{\|T(x)\|\|T(y)\|}\right) = \cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right)$ for all $x, y \in \mathbb{R}^n$

The next result establishes that inner product preserving and the norm preserving linear maps on $\mathbb{R}^n$ are the same.

Proposition 0.2. *A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is inner product preserving if and only if it is norm preserving.*

Proof. Let T be norm preserving. Then for all $x, y \in \mathbb{R}^n$ we have the following

$$\langle T(x + y), T(x + y)\rangle = \langle x + y, x + y\rangle$$
$$\Rightarrow \|T(x)\|^2 + \|T(y)\|^2 + 2\langle T(x), T(y)\rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$$
$$\Rightarrow \langle T(x), T(y)\rangle = \langle x, y\rangle,\ (\because \|T(x)\| = \|x\|)$$

which proves that T is inner product preserving.
Conversely, if T is inner product preserving then $\langle T(x), T(x)\rangle = \langle x, x\rangle$ which proves that $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$ and that T is norm preserving. □

Remark: It is easy to see that the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is norm preserving or inner product preserving, then it is angle preserving as well.

Proposition 0.3. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then there is a positive real number M such that $\|T(h)\| \leq M\|h\|$ for all $h \in \mathbb{R}^m$.*

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Proof. Let $A = (a_{ij})_{m\times n}$ be matrix of T with respect to the standard basis in $\mathbb{R}^n$. Then

$$T(h) = Ah = \left(\sum_{j=1}^{m} a_{1j}h_j, \cdots, \sum_{j=1}^{m} a_{nj}h_j\right)^t.$$

Therefore if we let $a_i = (a_{i1}, ..., a_{i,n})^t$ then

$$\begin{aligned}\|T(h)\|^2 &= \sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}h_j\right)^2 \\ &= \sum_{i=1}^{n}\langle a_i, h\rangle^2 \leq \sum_{i=1}^{n}\|a_i\|^2\|h\|^2\end{aligned}$$

where the last step has been obtained using the Cauchy-Schwarz inequality. Defining $M := \sqrt{(\sum_{i=1}^n \|a_i\|^2)}$ gives the desired result. □

Corollary 0.4. *Every linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuous in the standard topology.*

Proof. For any $\epsilon > 0$ and $x, y \in \mathbb{R}^n$, consider using the proposition 0.3

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq M\|x - y\| < \epsilon$$

for $\|x - y\| < \delta := \frac{\epsilon}{M}$, which proves continuity of T. □

Proposition 0.5. *Let $(\mathbb{R}^n)^*$ denote the dual space of $\mathbb{R}^n$. Then the map $\Phi : \mathbb{R}^n \to (\mathbb{R}^n)^*$ defined by*

$$\Phi(x)(y) = \varphi_x(y) = \langle x, y\rangle\,,\ \forall\ x, y \in \mathbb{R}^n$$

is injective. Moreover, every $\varphi \in (\mathbb{R}^n)^*$ is such that $\varphi = \varphi_x$ for some $x \in \mathbb{R}^n$.

Proof. Let $\Phi(x_1) = \Phi(x_2)$. Then $\varphi_{x_1}(y) = \varphi_{x_2}(y)$ for all $y \in \mathbb{R}^n$. This means

$$\langle x_1, y\rangle = \langle x_2, y\rangle$$

from which we see that $\langle x_1 - x_2, y\rangle = 0$ which is possible for all y if and only if $x_1 - x_2 = 0$. This proves injectivity.
Now let $\varphi \in (\mathbb{R}^n)^*$. Take the standard basis $B := \{e_1, ..., e_n\}$ of $\mathbb{R}^n$. Consider for $x_1, x_2 \in \mathbb{R}^n$

$$\Phi(x_1 + x_2)(y) = \langle x_1 + x_2, y\rangle = (\Phi(x_1) + \Phi(x_2))(y).$$

Also for any scalar α

$$\Phi(\alpha x)(y) = \langle \alpha x, y\rangle = (\alpha\Phi(x))(y).$$

This proves that Φ is a linear map. Therefore it preserves the basis also (since the two vector spaces $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ are linearly isomorphic!). This means $\Phi(B)$ is a basis of $(\mathbb{R}^n)^*$ and thus

$$\varphi = \alpha_1\Phi(e_1) + \cdots + \alpha_n\Phi(e_n) = \Phi(\alpha) = \varphi_\alpha$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$. □

Remark: In the above proof, we have used the equality of two maps. Two maps $f : A \to B$ and $g : A \to B$ are said to be equal if $f(x) = g(x)$ for all $x \in A$. We have also used the vector space structure of the dual space $(\mathbb{R}^n)^*$.

0.2 References:

1. M. Spivak. *Calculus on Manifold*, Addison Wesley (1965)

Exercises

1. Let $x_1, ..., x_n$ be a basis of $\mathbb{R}^n$ and $\lambda_1, ..., \lambda_n$ be scalars such that $T(x_i) = \lambda_i x_i$, prove that if all λ_i are equal then T is angle preserving.

2. If $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, show that for any $\theta_1, \theta_2 \in \mathbb{R}$, $R_{\theta_1}R_{\theta_2} = R_{\theta_1+\theta_2}$. Show that R_θ is angle preserving and that for $x \neq 0$ angle between x and $R_\theta(x)$ is θ.

3. Prove the following for $x, y \in \mathbb{R}^n$:

   (a) $\|x + y\| \leq \|x\| + \|y\|$

   (b) $\|x - y\| \leq \|x\| + \|y\|$

   (c) $\|x - y\| \geq \|x\| - \|y\|$

   (d) $|\|x\| - \|y\|| \leq \|x - y\|$

4. if $x \perp y$ then show that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

5. Prove the triangle inequality: $\|x - y\| \leq \|x - z\| + \|z - y\|$ for all $x, y, z \in \mathbb{R}^n$.

6. Given nonnegative real number $\alpha_1, ..., \alpha_n$, prove the arithmetic and geometric mean inequality:

$$\alpha_1\alpha_2\cdots\alpha_n \leq \left(\frac{\alpha_1 + \cdots + \alpha_n}{n}\right)^n.$$