Department of Mathematics
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Real Analysis I
Lecture notes on 'functions of bounded variation'

Definition: By a partition of interval $[a,b]$, $a<b$, $a,b,\in\mathbb{R}$, we mean a finite-collection $\pi=\{[t_{i-1},t_i]\mid i=1,...,n\}$, $n\in\mathbb{N}$ of closed intervals such that $t_0=a<t_1<\cdots<t_{n-1}<t_n=b$. We some time represent π by $t_0=a<t_1<\cdots<t_{n-1}<t_n=b$.

Definition: Let $f:[a,b]\to\mathbb{R}$ and $a=t_0<t_1<\cdots<t_k=b$ be a partition π of $[a,b]$. Let $\Delta f_i:=f(t_i)-f(t_{i-1})$ for each $i=1,..,k$. Define

$$
\begin{aligned}
p&=\sum_{i=1}^{k}\frac{\Delta f_i}{2}\left(1+\frac{\Delta f_i}{|\Delta f_i|}\right),\ P_a^b(f)=\sup_{\text{all }\pi}\{p\}\\
n&=-\sum_{i=1}^{k}\frac{\Delta f_i}{2}\left(1-\frac{\Delta f_i}{|\Delta f_i|}\right),\ N_a^b(f)=\sup_{\text{all }\pi}\{n\}\\
t&=\sum_{i=1}^{k}|\Delta f_i|,\ T_a^b(f)=\sup_{\text{all }\pi}\{t\}
\end{aligned}
\tag{0.1}
$$

$P_a^b(f)$, $N_a^b(f)$, and $T_a^b(f)$ are called positive, negative, and total variations of f over $[a,b]$. If $T\in\mathbb{R}$, we say that f is of **bounded variation** over $[a,b]$ and we write this as $f\varepsilon BV$.

Remark: The concept of bounded variation makes sense only for compact intervals!

Note that $t=p+n$ and $f(b)-f(a)=p-n$. Also, $p,n,t\geq 0$. Therefore, $P_a^b(f)\leq T_a^b(f)\leq P_a^b(f)+N_a^b(f)$.

Example: A constant function is of bounded variation over $[a,b]$ with $T=0$.

Example: A monotonically increasing function f is of bounded variation over $[a,b]$ Since here $T_a^b(f)=f(b)-f(a)$.

Example: Let $f:[0,1]\to\mathbb{R}$ be

$$
f(x)=\begin{cases}\sqrt{x}\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0,\\ 0 & \text{if } x=0;\end{cases}
$$

Then for the partition consisting of $(0,\frac{2}{(2N+1)\pi})\cup\{(\frac{2}{(n+1)\pi},\frac{2}{n\pi})\mid n=1,...,2N\}\cup(\frac{2}{\pi},1)$ we have

$$
\begin{aligned}
t&=\sum_{i=1}^{2N+1}\sqrt{\frac{2}{n\pi}}+\left|\sin 1-\sqrt{\frac{2}{\pi}}\right|\\
&\geq\sqrt{\frac{2}{\pi}}\sum_{i=1}^{2N+1}\frac{1}{n}=\sqrt{\frac{2}{\pi}}H_{2N+1}
\end{aligned}
$$

where H_k is the $k-$th harmonic number. Since $H_k\to\infty$ as $k\to\infty$, it follows that t is not bounded above in $\mathbb{R}$. Therefore, the given f is not of bounded variation over $[0,1]$.

Lemma 0.1. *Let $f:[a,b]\to\mathbb{R}$ is monotonically increasing and is discontinuous at some $c\in[a,b]$. Then $\lim_{x\to c^+}f(x)$ and $\lim_{x\to c^-}f(x)$ always exist.*

Proof. Observe that f is bounded above by $f(c)$ on $[a,c]$. Therefore, $\alpha:=\sup_{x\in(a,c)}\{f(x)\}\leq f(c)$. So, for all $t\in(a,c)$ we have $f(a)\leq f(t)\leq\alpha\leq f(c)$. Now for given $\epsilon>0$, $\alpha-\epsilon$ is not an upper bound of the set $f((a,c))$. Therefore, there is a $f(t_0)\in f((a,c))$ for some $a<t_0<c$ such that $\alpha-\epsilon<f(t_0)<f(c)$. Define $\delta:=c-t_0$. Then for all $c-\delta<t<c$ implies $\alpha-\epsilon<f(c-\delta)\leq f(t)\leq f(c)$ since f is increasing. This proves $\lim_{x\to c^-}f(x)=\alpha$. The other part of the proof can be dealt the same way for α replaced by $\beta=\inf_{x\in(c,b)}\{f(x)\}$. □

Lemma 0.2. *If f is of bounded variation on $[a,b]$, then $T_a^b(f)=P_a^b(f)+N_a^b(f)$ and $f(b)-f(a)=P_a^b(f)-N_a^b(f)$.*

Proof. For any partition π of $[a,b]$, $p=n+f(b)-f(a)$, which gives $\sup_{\text{all}\pi}p=\sup_{\text{all}\pi}[n+f(b)-f(a)]=\sup_{\text{all}\pi}n+f(b)-f(a)$. This proves $f(b)-f(a)=P_a^b(f)-N_a^b(f)$.

For the first part, we consider $t=p+n=2p-[f(b)-f(a)]$ from which it follows that,
$T_a^b(f)=2P_a^b(f)-[f(b)-f(a)]=P_a^b(f)+N_a^b(f)$. □

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Theorem 0.3. *Let $f, g \varepsilon BV$ over the interval $[a,b]$ and c is a constant. Then*

1. *$f \varepsilon BV$ over every closed subinterval of $[a,b]$.*
2. *f is bounded on $[a,b]$*
3. *$cf,\ fg,\ (f \pm g) \varepsilon BV$ over $[a,b]$*
4. *$1/g,\ (f/g) \varepsilon BV$ provided $1/g$ is bounded on $[a,b]$.*

Proof. 1. Let $x, y \in [a,b]$, such that $x < y$ and π be a partition of $[x,y]$. Then $\pi_1 = \pi \cup \{(a,x),(y,b)\}$ of $[a,b]$ is a partition of $[a,b]$ such that $t = \sum_{i=1}^{k} |\Delta f_i| \leq \sum_{i=1}^{k} |\Delta f_i| + |f(x) - f(a)| + |f(b) - f(y)| \leq T_a^b(f)$. Thus $T_x^y(f) \leq T_a^b(f)$. So, $f \varepsilon BV$ over $[x,y]$.

2. For any $x \in [a,b]$, by Lemma 0.2 $f(x) = f(a) + P_a^x(f) - N_a^x(f) \leq f(a) + P_a^b(f)$, since $P_a^x(f)$ and $N_a^x(f)$ are non-negative increasing functions of x. Thus f is bounded on $[a,b]$.

3. Since $\sup\{|c|t\} = |c| \sup\{t\}$, so $cf \varepsilon BV$.

For a partition π of $[a,b]$ consider $t = \sum_{i=1}^{k} |\Delta(f \pm g)_i| \leq \sum_{i=1}^{k} |\Delta f_i| + \sum_{i=1}^{k} |\Delta g_i| \leq T_a^b(f) + T_a^b(g)$. Thus $T_a^b(f \pm g) \leq T_a^b(f) + T_a^b(g)$ which proves $f \pm g \varepsilon BV$.

4. Let $1/|g(x)| \leq M$ for some $M > 0$ and all $x \in [a,b]$. Consider for any partition of $[a,b]$, $t = \sum_{i=1}^{k} \left| \frac{1}{g(t_i)} - \frac{1}{g(t_{i-1})} \right| \leq M^2 \sum_{i=1}^{k} |\Delta g_i| \leq M^2 T_a^b(g)$. Thus $(1/g) \varepsilon BV$ and using 3, $(f/g) \varepsilon BV$. □

The next result characterizes the functions of bounded variations.

Theorem 0.4. *A function f is of bounded variation over $[a,b]$ if and only if f is the difference of two monotone real-valued functions on $[a,b]$.*

Proof. If $f = g - h$ for some g and h increasing on $[a,b]$, then $t \leq \sum_{i=1}^{k} \Delta g_i + \sum_{i=1}^{k} \Delta h_i = g(b) - g(a) + h(b) - h(a)$, which proves t is bounded above in $\mathbb{R}$. So, $T_a^b(f) \in \mathbb{R}$.

Conversely, if f is of bounded variation over $[a,b]$, then define $g(x) = P_a^x(f)$ and $h(x) = N_a^x(f)$ such that $f(x) - f(a) = P_a^x(f) - N_a^x(f)$ by Lemma 0.2. Then g and h are monotonically increasing functions and are real valued since $0 \leq P_a^x(f) \leq T_a^x(f) \leq T_a^b(f) \in \mathbb{R}$, and similarly $N_a^x(f) \in \mathbb{R}$. This completes the proof. □

Corollary 0.5. *If $f \varepsilon BV$ over $[a,b]$ then for each point of discontinuity $c \in (a,b)$, $\lim_{x \to c^{\pm}} f(x)$ exist.*

Proof. By the preceding theorem $f = g - h$ where g and h are monotonically increasing on $[a,b]$. The result follows now from Lemma 0.1. □

Theorem 0.6. *Let $f : [a,b] \to \mathbb{R}$ such that $f \varepsilon BV$. Let $F : [a,b] \to \mathbb{R}_0$ by*

$$F(x) := \begin{cases} 0 & \text{if } x = a, \\ T_a^x(f) & \text{if } x \neq a; \end{cases}$$

Then

1. *$F(y) - F(x) = T_x^y(f)$ for all $x, y \in [a,b]$ such that $x < y$*
2. *F is increasing on $[a,b]$*
3. *F is continuous at $x \in [a,b]$ if and only if so is f*

Proof. Since $T_a^y(f) - T_a^x(f) = T_x^y(f)$ for all $x < y$, 1 and 2 follow.

3. If F is continuous at $x \in [a,b]$, then for any $\epsilon > 0$, choose $\delta > 0$ such that $|F(y) - F(x)| < \epsilon$ for all $|y - x| < \delta$. This gives on $x - \delta < y < x$, $T_y^x < \epsilon$, i.e., $\sum_{i=1}^{k} |\Delta f_i| < \epsilon$ for a partition of $[x,y]$ of size k. For this δ, $f(x) = f(y) + P_y^x(f) - N_y^x(f)$ which gives $|f(x) - f(y)| \leq P_y^x(f) \leq T_y^x(f) < \epsilon$. Thus $\lim_{y \to x^-} f(x) = f(x)$. Similarly the righthand limit of f at x is $f(x)$. This proves continuity of f at x.

Conversely, let f is continuous at $x \in [a,b]$ and for a given ϵ, we have $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \frac{\epsilon}{2}$. Also there is a partition π defined by the points $p_0, ..., p_k$ of $[x,b]$ such that $T_x^b(f) < t + \frac{\epsilon}{2}$ where $t = \sum_{i=1}^{k} |f(p_i) - f(p_{i-1})|$. We construct a partition π_1 of $[x,b]$ from π as follows. If $x \in (p_1 - \delta, p_1 + \delta)$ then $\pi_1 := \pi$ otherwise take a point $x_1 \in (x, p_1)$ such that $x_1 - x < \delta$ and define π_1 corresponding to $q_0, q_1, q_2 ..., q_{k+1}$ such that $q_0 = p_0,\ q_1 = x_1,\ \ p_i = q_{i+1}$ for all $i > 1$. Now consider for $q_0 = x < y < x + \delta$, $|F(y) - F(x)| = T_x^y(f) = T_x^b(f) - T_y^b(f) < t + \frac{\epsilon}{2} - T_y^b(f) \leq t + \frac{\epsilon}{2} - T_{q_1}^b \leq$

$\sum_{i=1}^{k+1} |f(q_i) - f(q_{i-1})| + \frac{\epsilon}{2} - \sum_{i=2}^{k+1} |f(q_i) - f(q_{i-1})|$ $= |f(q_1) - f(x)| + \frac{\epsilon}{2} < \epsilon$ for $|f(x) - f(y)| < \frac{\epsilon}{2}$. This proves that $\lim_{y \to x^+} F(x) = F(x)$. The other limit can be dealt the same way. □

Definition: (**absolute continuity**). $f : [a, b] \to \mathbb{R}$ is said to be absolutely continuous on $[a, b]$ if, for given $\epsilon > 0$, there is a $\delta > 0$ such that $\sum_{i=1}^{n} |f(x_i) - f(t_i)| < \epsilon$ for every finite collection $\{(t_i, x_i)\}_{i=1}^{n}$ of non-overlapping intervals with $\sum_{i=1}^{n} |x_i - t_i| < \delta$.

Example: An absolutely continuous function f on $[a, b]$ is of bounded variation over $[a, b]$. To see this, let δ be as in the definition of absolutely continuous function for $\epsilon = 1$. Then each partition of $[a, b]$ is a finite set of non-overlapping intervals, which can be further decomposed into K sets of subintervals each of length less than δ. Then $K \leq (b - a)/\delta$. Thus for any such subdivision $t \leq (1 + 1 \cdots + 1) - K$ times $= K \leq (b - a)/\delta$ which proves $T \in \mathbb{R}$. Thus $f \varepsilon BV$

1. Let A and B be subsets of $\mathbb{R}$ which are bounded above. Let $A + B := \{a + b \mid a \in A,\ b \in B\}$. Now prove the following
   (a) if $A \subseteq B$, then $\sup A \leq \sup B$
   (b) $A + B$ is bounded above in $\mathbb{R}$ and that $\sup(A + B) \leq \sup A + \sup B$
   (c) For any real $c \neq 0$, if $c > 0$ then $\sup(cA) = c \sup A$ and if $c < 0$ then cA is bounded below and $\inf(cA) = c \sup(A)$.
2. Prove that if $f \varepsilon BV$ such that $T_a^b(f) = 0$, then f is constant on $[a, b]$.
3. If $f \varepsilon BV$ on $[a, b]$, prove that for each $c \in (a, b)$, $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist. Also prove that the set of discontinuities of f is countable.
4. Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ are of bounded variation on $[a, b]$ and $[c, d]$ respectively. Check whether $g \circ f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$?