Department of Mathematics

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Assignment #1: August 16, 2010

B.Tech.(ECE) Sem. I, Sec. A, B

Matrices

1. Prove the following:
   (a) A $m \times n$ matrix A with real entries is a linear map from $\mathbb{R}^n \to \mathbb{R}^m$.
   (b) For square matrices A and B each of order n, $(AB)^T = B^T A^T$. Further if A and B are invertible then $(AB)^{-1} = B^{-1}A^{-1}$.
   (c) For a square matrix $A = (a_{ij})_{n\times n}$, $\sum_{j=1}^{n} C_{ij}a_{ij} = \det(A)$ for all $1 \le i \le n$ where C_{ij} dentes the cofactor of a_{ij}.
   (d) Every square matrix A can be written as a sum of symmetric matrix and a skew symmetric matrix.
   (e) For some integer $n > 1$, if λ is an eigenvalue of a $n \times n$ matrix A then $\det(A - \lambda I) = 0$. Further if X is an eigenvector associated with the eigenvalue λ then $(A - \lambda I) \cdot X = O$. Here I denotes identity matrix of order n.
   (f) Product of eigenvalues of a square matrix A is equal to its determinant.
   (g) If all the eigenvalues of a square matrix are distinct then it is diagonalizable.
   (h) Product of any two orthogonal matrices is an orthogonal matrix. The set of all $n \times n$ orthogonal matrices under matrix multiplication is called the *Orthogonal group.*
   (i) Product of any two unitary matrices is an unitary matrix. The set of all $n \times n$ unitary matrices under matrix multiplication is called the *Unitary group.*
   (j) Eigenvalues of a Hermitian matrix are always real.
   (k) Eigenvalues of a Unitary matrix always lie on the unit circle in the complex plane.
   (l) If λ is an eigenvalue of an invertible matrix A then λ^{-1} is the eigenvalue of A^{-1}.

2. Determine a, b, c if the matrix $A = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$ is orthogonal.

3. State and prove Cayley-Hamilton theorem. Using it evaluate the inverse of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

4. Find eigenvalues and eigenvectors of the following matrices and check their diagonalizability.
   (i) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ (v) $\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

5. Calculate A^{-3} and A^4 if $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}$.

6. Reduce the quadratic forms (i) $6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy$ (ii) $5x^2 + 26y^2 + 10z^2 + 4yz + 14zx + 6xy$ (iii) $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4xy$ to their canonical forms. Also obtain the underlying transformation rules in each case.

7. Define a bilinear form. For a square matrix A of order n show that the map $f : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by $f(X, Y) := X^T A\overline{Y}$ where $\overline{Y}$ denotes the matrix obtained by replacing each of the entries of Y by its complex conjugate, is a bilinear form.

8. Prove that the matrix $A = \frac{1}{2}\begin{pmatrix} 1+\iota & -1+\iota \\ 1+\iota & 1-\iota \end{pmatrix}$ is unitary. Obtain the associated bilinear form of this matrix.

9. If $S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix}$ where $a = e^{\iota 2\pi/3}$ show that $S^{-1} = \frac{1}{3}\overline{S}$.

10. Let $X = \begin{pmatrix} x & y & z \end{pmatrix}^T$; $X' = \begin{pmatrix} x' & y' & z' \end{pmatrix}^T$ and $O = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \end{pmatrix}$, $\theta \in [-\pi/2, \pi/2], \varphi \in [0, 2\pi]$. Show that the linear transformation $X' = OX$ is orthogonal. Obtain the inverse transformation rule.

11. Prove that every Hermitian matrix can be written as $A + \iota B$ where A is real symmetric matrix and B is a real skew-symmetric matrix. Prove further that every square matrix can be written as $P + \iota Q$ where P and Q are Hermitian matrices.

12. Give an example of a matrix which is not diagonalizable.

13. What do you mean by linearly dependent and independent vectors in the vector space $\mathbb{R}^n$ over reals ?