Solutions of the Bessel's Equation

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Abstract

In these notes we discuss the solutions of the Bessel equation $L(y) = 0$ where the linear map

$$L(y) := x^2y'' + xy' + (x^2 - \alpha^2)y. \tag{0.1}$$

Here $a,\ b$ have convergent power series expansion in some open interval $|x| < r_0$ and α a scalar. (see Coddington [1]).

Recall that a point $x_0 \in I$ is said to be a regular singular point of a second order ODE if it can be put in the form $L(y) = (x - x_0)^2y'' + a(x)(x - x_0)y' + b(x)y = 0$ in a neighborhood of $x = x_0$ such that the functions a and b are analytic at x_0. As an example, observe that the point $x = 0$ is a regular singular point of the Bessel's equation (Eq.(0.1)).

The Euler Equation

Consider the Euler equation

$$L(y) := x^2y'' + axy' + by = 0,\ x > 0 \tag{0.2}$$

where $a,\ b$ are constant. Then for a scalar r $L(x^r) = q(r)x^r$ where

$$q(r) := r(r - 1) + ar + b \tag{0.3}$$

is the indicial polynomial for the Eq. (0.2). If r_1, r_2 are roots of q then $L(x_1^r) = 0 = L(x_2^r)$. If $r_1 \neq r_2$ then $\{x^{r_1},\ x^{r_2}$ form basis for the solution space of Eq. (0.2) on $x > 0$. On the other hand if $r_1 = r_2$ then $q(r) = (r - r_1)^2$ and thus $q'(r_1) = 0$. Let us denote $\frac{\partial}{\partial r} \equiv D$ and observe that

$$DL(x^r) = LD(x^r)$$

which gives us $L(x^r \log(x)) = D(q(r)x^r) = q'(r)x^r + q(r)x^r \log(x)$. Therefore $L(x^{r_1} \log(x)) = (q'(r_1) + q(r_1))x^{r_1} = 0$. Hence for $r_1 = r_2$ the set $\{x^{r_1},\ x^{r_1} \log(x)\}$ form a basis for the solution space of Eq. (0.2). These solution sets can be extended to the case when $x < 0$ just by changing $x \to -x$ and we see that the basis for the solution of the Eq. (0.2) exist at all $x \neq 0$. We have proved the following theorem.

Theorem 1. A basis $\{\varphi_1, \varphi_2\}$ for the solution space of Euler's equation on $\mathbb{R} - \{0\}$ is given by

$$\varphi_1, \varphi_2 = \begin{cases} |x|^{r_1},\ |x|^{r_2} & \text{if } r_1 \neq r_2, \\ |x|^{r_1},\ |x|^{r_1} \log|x| & \text{if } r_1 = r_2 \end{cases} \tag{0.4}$$

We now turn our attention towards the case when a and b in Euler equation are not constants but have power series expansion about the point $x = 0$ in $|x| < r_0$ then the theorem 1 generalizes to the next theorem 2 (see [1] for proof).

Theorem 2. A basis $\{\varphi_1, \varphi_2\}$ for the solution space of the equation $x^2y'' + a(x)xy + b(x)y = 0$, where $a,\ b$ have convergent power series expansion for $0 < |x| < r_0$ and $r_1,\ r_2$ (real(r_1) $\geq$ real(r_2)) are roots of the indicial polynomial $q(r) = r(r-1) + a(0)r + b(0)$, is given by

$$\varphi_1, \varphi_2 = \begin{cases} |x|^{r_1}\sum_{k=0}^{\infty} c_k x^k (c_0 = 1),\ |x|^{r_2}\sum_{k=0}^{\infty} C_k x^k (C_0 = 1) & \text{if } r_1 - r_2 \neq \{0\} \cup \mathbb{Z}^+, \\ |x|^{r_1}\sigma_1(x),\ |x|^{r_1+1}\sigma_2(x) + |x|^{r_1}\sigma_1(x)\log|x| & \text{if } r_1 - r_2 = 0,\ \sigma_1(0) \neq 0, \\ |x|^{r_1}\sigma_1(x),\ |x|^{r_2}\sigma_2(x) + c|x|^{r_1}\sigma_1(x)\log|x| & \text{if } r_1 - r_2 = \mathbb{Z}^+,\ \sigma_1(0) \neq 0 \neq \sigma_2(0), \end{cases}$$

where $\sigma_1,\ \sigma_2$ have convergent power series expansion in $|x| < r_0$ and c is a constant.

The Bessel Equation

Now we apply the above theory to obtain the solutions of the Bessel equation

$$L(y) = x^2y'' + xy' + (x^2 - \alpha^2)y = 0 \tag{0.5}$$

where α with real(α) ≥ 0 is a scalar. Comparing it with the differential equation in theorem 2 above, here $a(x) = 1$ and $b(x) = x^2 - \alpha^2$ which are real analytic everywhere on $\mathbb{R}$. Also $q(r) = r(r-1) + r - \alpha^2 = r^2 - \alpha^2$ whose roots are $r_1 = \alpha,\ r_2 = -\alpha$ which are always distinct for $\alpha \neq 0$. Applying the theorem 2 above we have the following basis for solution space of Eq. (0.5) on $0 < |x| < r_0$,

$$\varphi_1, \varphi_2 = \begin{cases} |x|^{\alpha}\sum_{k=0}^{\infty} c_k x^k (c_0 = 1),\ |x|^{-\alpha}\sum_{k=0}^{\infty} C_k x^k (C_0 = 1) & \text{if } 2\alpha \notin \{0\} \cup \mathbb{Z}^+, \\ \sum_{k=0}^{\infty} c_k x^k (c_0 \neq 0),\ |x|\sigma_2(x) + \varphi_1(x)\log|x| & \text{if } \alpha = 0, \\ |x|^{r_1}\sum_{k=0}^{\infty} c_k x^k (c_0 \neq 0),\ |x|^{r_2}\sigma_2(x) + c\varphi_1(x)\log|x| & \text{if } 2\alpha \in \mathbb{Z}^+, \end{cases}$$

or for $x > 0$

$$\varphi_1, \varphi_2 = \begin{cases} J_\alpha(x),\ J_{-\alpha}(x) & \text{if } 2\alpha \notin \{0\} \cup \mathbb{Z}^+, \\ J_0(x),\ K_0(x) & \text{if } \alpha = 0, \\ J_\alpha(x),\ K_\alpha(x) & \text{if } 2\alpha \in \mathbb{Z}^+, \end{cases}$$

where

$$J_0(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m},\quad K_0(x) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} H_m \left(\frac{x}{2}\right)^{2m} + J_0(x)\log(x)$$

$$J_\alpha(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1+m+\alpha)}\left(\frac{x}{2}\right)^{2m+\alpha},$$

$$K_\alpha(x) := -\frac{1}{2}\sum_{j=0}^{\alpha-1} \frac{\alpha-1-j}{j!}\left(\frac{x}{2}\right)^{2j-\alpha} - \frac{H_\alpha}{2\alpha!}\left(\frac{x}{2}\right)^{\alpha} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{p=0}^{\alpha} \frac{(-1)^m}{m!(m+\alpha)!} H_{m+p}\left(\frac{x}{2}\right)^{2m+\alpha} + J_\alpha(x)\log(x)$$

The symbol H_m is given by

$$H_m := 1 + \frac{1}{2} + \cdots + \frac{1}{m}.$$

$J_\alpha(x)$ and $K_\alpha(x)$ are called as the Bessel functions of order α and of first and second kind respectively.

Recurrence Relations and Orthogonal Properties

The Bessel function of first kind

$$J_\alpha(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1+m+\alpha)} \left(\frac{x}{2}\right)^{2m+\alpha}, \ x > 0$$

is a smooth function of x and satisfies the following

$$\begin{aligned}
\left(x^\alpha J_\alpha(x)\right)' =& 2^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1+m+\alpha)} \frac{d}{dx} \left(\frac{x}{2}\right)^{2m+2\alpha} \\
=& 2^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1+m+\alpha)} (m+\alpha) \left(\frac{x}{2}\right)^{2m+2\alpha-1} \\
=& 2^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+\alpha)\Gamma(m+\alpha)} (m+\alpha) \left(\frac{x}{2}\right)^{2m+2\alpha-1} \\
=& x^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha)} \left(\frac{x}{2}\right)^{2m+\alpha-1} = x^\alpha J_{\alpha-1}(x)
\end{aligned} \tag{0.6}$$

which gives the identity

$$x^\alpha J_\alpha'(x) + \alpha x^{\alpha-1} J_\alpha(x) = x^\alpha J_{\alpha-1}(x) \quad \text{or} \quad x J_\alpha'(x) + \alpha J_\alpha(x) = x J_{\alpha-1}(x) \tag{0.7}$$

Similarly we have

$$\begin{aligned}
\left(x^{-\alpha} J_\alpha(x)\right)' =& 2^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1+m+\alpha)} \frac{d}{dx} \left(\frac{x}{2}\right)^{2m} \\
=& 2^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1+m+\alpha)} m \left(\frac{x}{2}\right)^{2m-1} \\
=& -x^{-\alpha} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(m-1)!\Gamma(1+m+\alpha)} \left(\frac{x}{2}\right)^{2m+\alpha-1} \\
=& -x^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(1+m+1+\alpha)} \left(\frac{x}{2}\right)^{2m+1+\alpha} = -x^{-\alpha} J_{\alpha+1}(x)
\end{aligned} \tag{0.8}$$

and this gives us

$$x^{-\alpha} J_\alpha'(x) - \alpha x^{-\alpha-1} J_\alpha(x) = -x^{-\alpha} J_{\alpha+1}(x) \quad \text{or} \quad x J_\alpha'(x) - \alpha J_\alpha(x) = -x J_{\alpha+1}(x) \tag{0.9}$$

On once adding and once subtracting (0.7) and (0.10) we obtain

$$\begin{aligned}
2J_\alpha'(x) =& J_{\alpha-1}(x) - J_{\alpha+1}(x) \\
2\alpha x^{-1} J_\alpha(x) =& J_{\alpha-1}(x) + J_{\alpha+1}(x)
\end{aligned} \tag{0.10}$$

which are the recurrence relations for the Bessel functions of first kind.

To obtain the orthogonal properties of $J_\alpha(x)$ we assume that $\alpha > 0$ and make a transformation $\psi_\lambda(x) = \sqrt{x} J_\alpha(\lambda x)$ for a scalar $\lambda > 0$ such that $0 \le x \le 1$ and $\psi_\lambda(0) = 0, \ \psi_\lambda(1) = J_\alpha(\lambda)$. Then for $\mu > 0$ and $\lambda > 0$, ψ_λ satisfies

$$\begin{aligned}
\psi_\lambda'' + \frac{1-4\alpha^2}{4x^2}\psi_\lambda = -\lambda^2 \psi_\lambda \\
\psi_\mu'' + \frac{1-4\alpha^2}{4x^2}\psi_\mu = -\mu^2 \psi_\mu
\end{aligned} \tag{0.11}$$

which after multiplying the first Eq. by ψ_μ and second one by ψ_λ and subtracting resulting latter Eq. from the former Eq. for $\lambda \neq \mu$ gives us

$$\psi_\mu \psi_\lambda'' - \psi_\mu'' \psi_\lambda = (\psi_\mu \psi_\lambda' - \psi_\mu' \psi_\lambda)' = (\mu^2 - \lambda^2)\psi_\lambda \psi_\mu \tag{0.12}$$

which on integrating on $[0,1]$ gives the following relation

$$\begin{aligned}(\psi_\mu \psi_\lambda' - \psi_\mu' \psi_\lambda)|_0^1 &= (\mu^2 - \lambda^2)\int_0^1 \psi_\lambda(x)\psi_\mu(x)dx = (\mu^2 - \lambda^2)\int_0^1 xJ_\alpha(\lambda x)J_\alpha(\mu x)dx \\ \Rightarrow \int_0^1 xJ_\alpha(\lambda x)J_\alpha(\mu x)dx &= \frac{1}{\mu^2 - \lambda^2}(\psi_\mu \psi_\lambda' - \psi_\mu' \psi_\lambda)|_0^1 = \frac{\lambda J_\alpha(\mu)J_\alpha'(\lambda) - \mu J_\alpha(\lambda)J_\alpha'(\mu)}{\mu^2 - \lambda^2}.\end{aligned} \tag{0.13}$$

This shows that if λ and μ are two positive zeros of $J_\alpha(x)$ i.e. $J_\alpha(\lambda) = 0 = J_\alpha(\mu)$ then

$$\int_0^1 xJ_\alpha(\lambda x)J_\alpha(\mu x)dx = 0.$$

Multiplying the Eq. $\psi_\lambda'' + \dfrac{1-4\alpha^2}{4x^2}\psi_\lambda = -\lambda^2\psi_\lambda$ throughout by ψ_λ and differentiating w.r.t. λ we obtain:

$$\begin{aligned} & \frac{d}{d\lambda}(\psi_\lambda \psi_\lambda'') + \frac{1-4\alpha^2}{4x^2}\frac{d}{d\lambda}(\psi_\lambda)^2 = -\frac{d}{d\lambda}\{\lambda^2(\psi_\lambda)^2\} \\ \Rightarrow\ & \frac{d\psi_\lambda}{d\lambda}\psi_\lambda'' + \psi_\lambda \frac{d\psi_\lambda''}{d\lambda} + \frac{1-4\alpha^2}{4x^2}2\psi_\lambda \frac{d\psi_\lambda}{d\lambda} = -2\lambda(\psi_\lambda)^2 - 2\lambda^2\psi_\lambda \frac{d\psi_\lambda}{d\lambda} \\ \Rightarrow\ & \frac{d\psi_\lambda}{d\lambda}\psi_\lambda'' + \psi_\lambda \frac{d\psi_\lambda''}{d\lambda} + 2\left(\frac{1-4\alpha^2}{4x^2} + \lambda^2\right)\psi_\lambda \frac{d\psi_\lambda}{d\lambda} = -2\lambda(\psi_\lambda)^2 \\ \Rightarrow\ & \frac{d\psi_\lambda}{d\lambda}\psi_\lambda'' + \psi_\lambda \frac{d\psi_\lambda''}{d\lambda} + 2\left(-\psi_\lambda''\right)\frac{d\psi_\lambda}{d\lambda} = -2\lambda(\psi_\lambda)^2 \\ \Rightarrow\ & \psi_\lambda \frac{d\psi_\lambda''}{d\lambda} - \psi_\lambda'' \frac{d\psi_\lambda}{d\lambda} = -2\lambda(\psi_\lambda)^2 \\ \Rightarrow\ & \left(\psi_\lambda \frac{d\psi_\lambda'}{d\lambda} - \psi_\lambda' \frac{d\psi_\lambda}{d\lambda}\right)' = -2\lambda(\psi_\lambda)^2 \end{aligned} \tag{0.14}$$

which on integrating gives

$$\begin{aligned} & \int_0^1 (\psi_\lambda(x))^2 dx = \int_0^1 (J_\alpha(\lambda x))^2 dx = -\frac{1}{2\lambda}\left(\psi_\lambda \frac{d\psi_\lambda'}{d\lambda} - \psi_\lambda' \frac{d\psi_\lambda}{d\lambda}\right)\Big|_0^1 \\ &= -\frac{1}{2\lambda}\left(J_\alpha(\lambda)\left\{\frac{J_\alpha'(\lambda)}{2} + \lambda J_\alpha''(\lambda) + J_\alpha'(\lambda)\right\} - \left\{\lambda J_\alpha'(\lambda) + \frac{J_\alpha'(\lambda)}{2}\right\}J_\alpha'(\lambda)\right) \\ &= -\frac{1}{2\lambda}\left(J_\alpha(\lambda)\left\{\lambda J_\alpha''(\lambda) + J_\alpha'(\lambda)\right\} - \left\{\lambda J_\alpha'(\lambda)\right\}J_\alpha'(\lambda)\right) \end{aligned} \tag{0.15}$$

which yields $\int_0^1 x\{J_\alpha(\lambda x)\}^2 dx = -\frac{1}{2\lambda}J_\alpha(\lambda)\left\{\lambda J_\alpha''(\lambda) + J_\alpha'(\lambda)\right\} + \frac{1}{2}\left\{J_\alpha'(\lambda)\right\}^2$. If we arrange the real roots of $J_\alpha(\lambda) = 0$ in a sequence $0 < \lambda_1 < \lambda_2 < \cdots$ then the set $\{J_\alpha(\lambda_n x)\}_{n\in\mathbb{Z}}^{+}$ is an orthogonal basis for the infinite dimensional Hilbert space $\mathcal{L}^2[0,1]$ of square integrable functions on $[0,1]$ with the inner product $< f(x), g(x) >:= \int_0^1 xf(x)g(x)dx$. Consequently each $f \in \mathcal{L}^2[0,1]$ is represented by $f(x) = \sum_{m=0}^{\infty} b_m J_\alpha(\lambda_m x)$ where $J_\alpha'(\lambda) \neq 0$, $b_m = \dfrac{2}{(J_\alpha'(\lambda_m))^2} < f(x), J_\alpha(\lambda_m x) >$.

Reference

[1] E. A. Coddington. *An introduction to ordinary differential equations*, Prentice Hall of India, 2004.